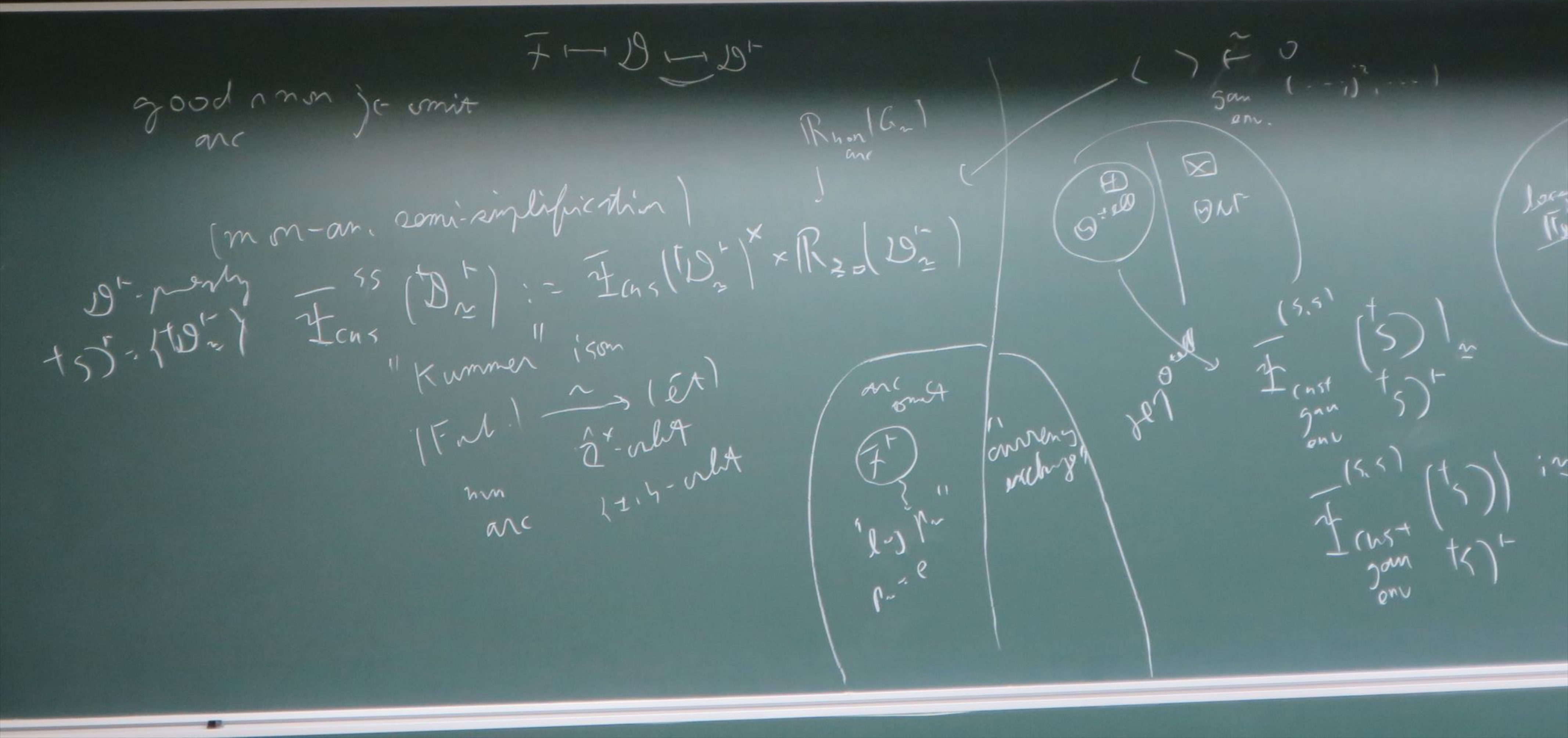


$\mathcal{F} \mapsto \vartheta \mapsto \vartheta^{\vdash}$

good non arc) ← omit

$\mathbb{R}_{\text{non arc}}(G_v)$

(non-arc. semi-simplification)

$\vartheta^{\vdash}$-prime-strip

${}^{\dagger}S)^{\vdash} = \{{}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}\}$

$\overline{\mathcal{I}}^{ss}_{cns}({}^{\dagger}\mathcal{D}^{\vdash}_{\simeq}) := \overline{\mathcal{I}}_{cns}({}^{\dagger}\mathcal{D}^{\vdash}_{\simeq})^{\times} \times \mathbb{R}_{\geq 0}(\mathcal{D}^{\vdash}_{\simeq})$

"Kummer" isom

$(\mathrm{Frob}) \xrightarrow{\sim} (\text{ét})$

$\hat{\mathbb{Q}}^{\times}$-orbit

non arc $\{\pm 1\}$-orbit

arc omit

$\mathcal{F}^{\vdash}$

"$\log p_v$"

$p_v = e$

"currency exchange"

$\langle\ \rangle \curvearrowleft \tilde{\ }\ 0$

gau env.

$(\ldots)^2, \ldots)$

$\boxplus$ $\Theta^{\pm ell}$

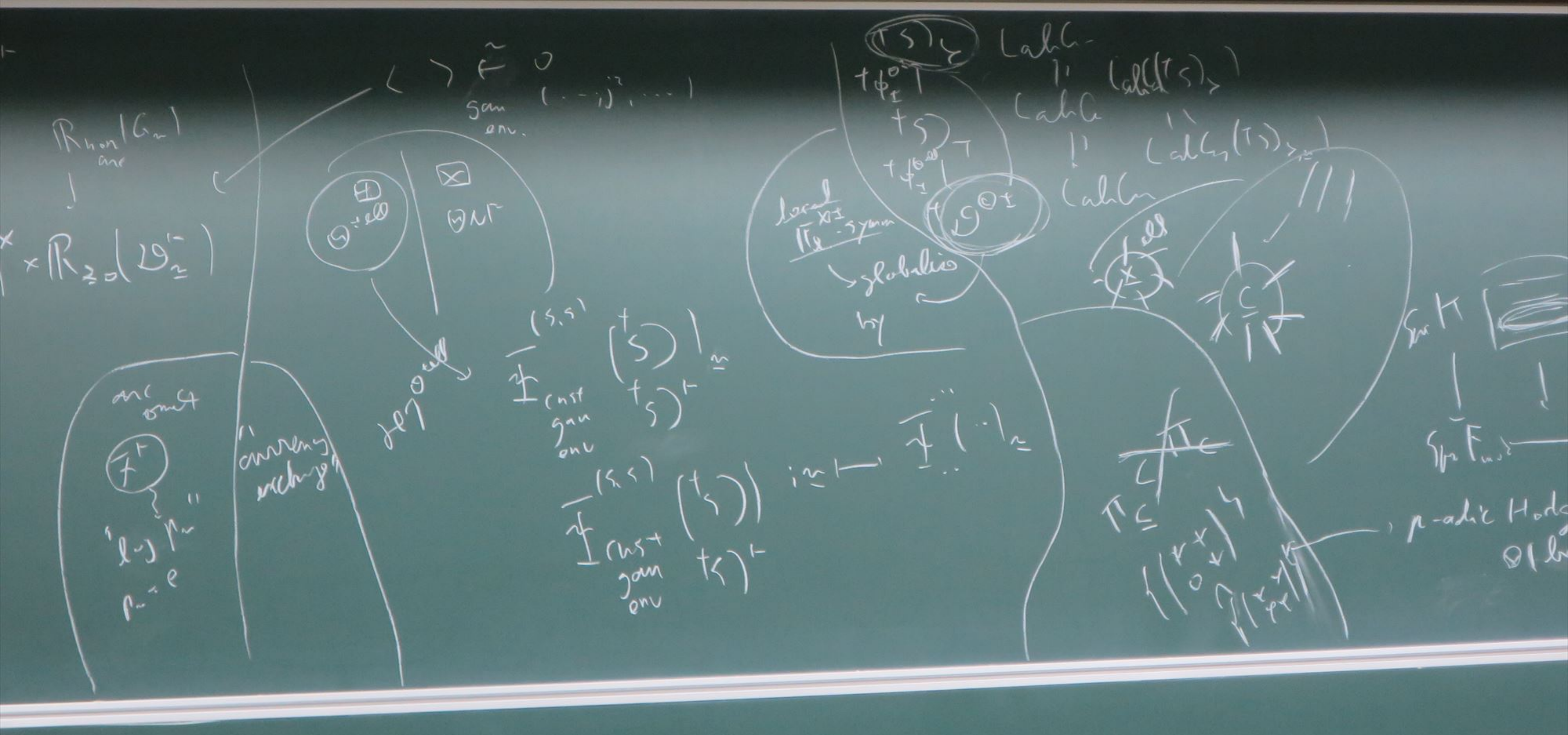
$\boxtimes$ ΘNF

poly $\Theta^{\pm ell}$

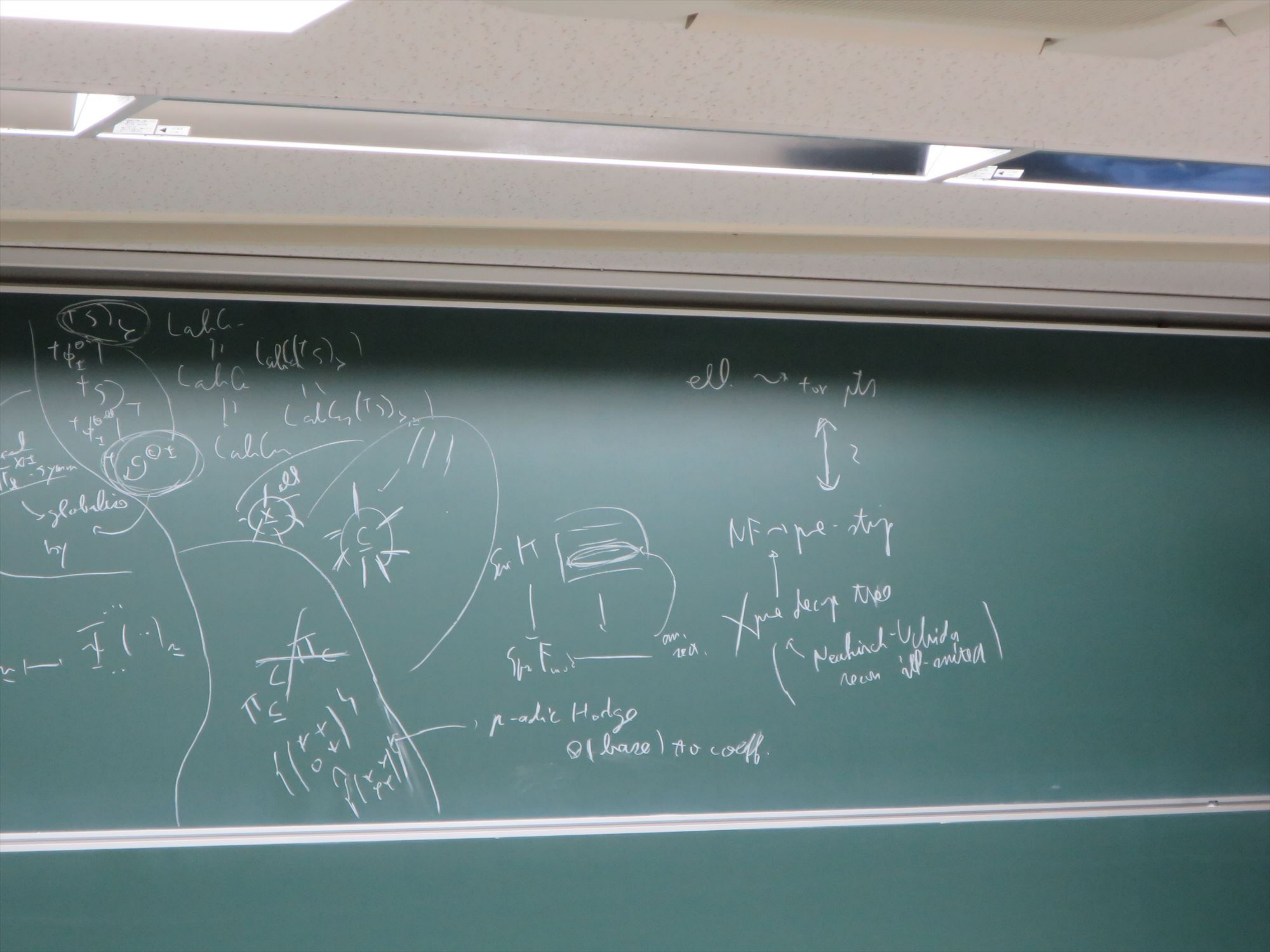
$\overline{\mathcal{I}}^{(s,s)}_{\text{cnst gau env}}({}^{\dagger}S)^{\vdash}_{\simeq}$

$\overline{\mathcal{I}}^{(s,s)}_{\text{cnst gau env}}({}^{\dagger}S)^{\vdash})$ in

loc



gau env.
currency exchange
const gau env
LabCusp
local
globaliz
p-adic Hodge
Spf K
Spf F_mod


LabCusp
LabCusp(†S)
LabCusp
LabCusp(†S)
LabCusp
globalize
by
ell.
⇝ tors pts
NF ⇝ pre-strip
X pre decomp trees
Neukirch-Uchida
recon
alg.-anitad
Spec K
Spec F_mod
ari.
red.
p-adic Hodge
⊗(base) to coeff.

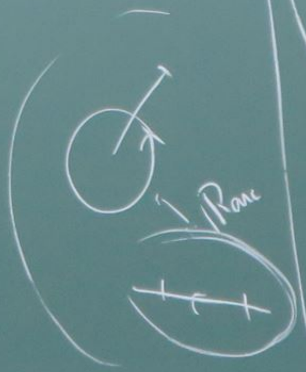
" $\mathcal{C}^{\Vdash}_{\mathrm{mod}}$ " (← Spec F_{mod})

$\boxed{+}$

$$^\dagger\mathfrak{S}^{\vdash} \longmapsto \mathcal{D}^{\Vdash}(^\dagger\mathfrak{S}^{\vdash})$$

w/ $\mathrm{Prime}(\mathcal{D}^{\Vdash}(^\dagger\mathfrak{S}^{\vdash})) \xrightarrow{\sim} \underline{\mathbb{V}}$

& $\{^\dagger\rho_{\mathcal{D}^{\Vdash},\underline{v}} : \underset{\text{gl.}}{\Phi_{\mathcal{D}^{\Vdash}(^\dagger\mathfrak{S}^{\vdash}),\underline{v}}} \xrightarrow{\sim} \underset{\text{loc}}{\mathbb{R}_{\geq 0}(^\dagger\mathfrak{S}^{\vdash}_{\underline{v}})}\}_{\underline{v}\in\underline{\mathbb{V}}}$



$\boxed{\times}$

[IUTc

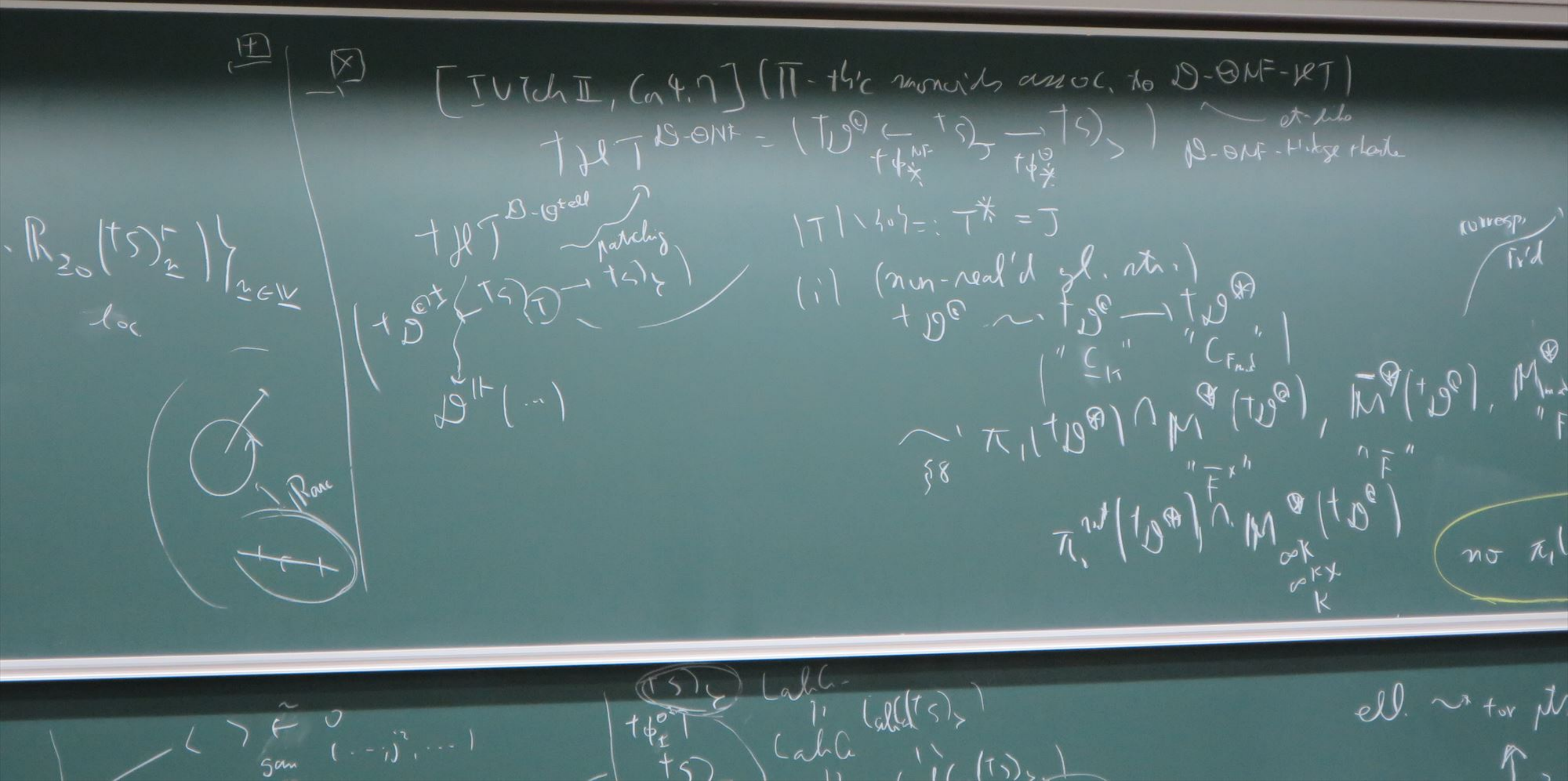
$^\dagger\mathcal{HT}^{\mathcal{D}\text{-}}$

$$\begin{tikzcd}(^\dagger\mathcal{D}^{\circledcirc\pm} & \{^\dagger\mathfrak{S}\} \arrow[l] \arrow[dl, squiggly] \\ \mathcal{D}^{\Vdash}(\ldots & \end{tikzcd}$$

$\mathcal{F} \longmapsto \mathcal{D} \longmapsto \mathcal{D}^{\vdash}$

$\langle\ \rangle\ \tilde{\mathcal{F}}$ $(\ldots;)^2, \ldots)$

sam



$\boxed{\pm}$ $\boxed{\times}$

$, \mathbb{R}_{\geq 0}(^\dagger\mathfrak{S})^{\Gamma}_{\underline{v}})\}_{\underline{v}\in\underline{\mathbb{V}}}$

loc

Rarc

[IUTch II, Cor 4.7] (Π-tic monoids assoc. to $\mathcal{D}$-ΘNF-$\mathcal{HT}$) ~ et-like $\mathcal{D}$-ΘNF-Hodge theater

$$^\dagger\mathcal{HT}^{\mathcal{D}\text{-}\Theta NF} = \left(^\dagger\mathcal{D}^{\circledcirc} \xleftarrow{^\dagger\phi^{NF}_{*}} {}^\dagger\mathfrak{S}_J \xrightarrow{^\dagger\phi^{\Theta}_{*}} {}^\dagger\mathfrak{S}_>\right)$$

$^\dagger\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm ell}}$ ~ patching

$$\left(^\dagger\mathcal{D}^{\odot\pm} \leftarrow {}^\dagger\mathfrak{S}_{T} \to {}^\dagger\mathfrak{S}_{\xi}\right)$$

$\mathcal{D}^{\vdash}(\ldots)$

$|T|\setminus\{0\} =: T^{*} = J$

(i) (non-real'd gl. rtn.)

$^\dagger\mathcal{D}^{\circledcirc} \rightsquigarrow {}^\dagger\mathcal{D}^{\circledcirc} \longrightarrow {}^\dagger\mathcal{D}^{\circledast}$

("$\underline{C}_K$" "$C_{F_{mod}}$")

$\underset{\S 8}{\rightsquigarrow}\ \pi_1(^\dagger\mathcal{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}(^\dagger\mathcal{D}^{\circledcirc}),\ \overline{\mathbb{M}}^{\circledast}(^\dagger\mathcal{D}^{\circledcirc}),\ \mathbb{M}^{\circledast}_{mod}$ "F

$\pi_1^{rat}(^\dagger\mathcal{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}_{\infty\kappa}(^\dagger\mathcal{D}^{\circledcirc})$ "$\overline{F}^{\times}$" "$\overline{F}$" $\infty\kappa\times$ κ

corresp. Fr'd

no $\pi_1($

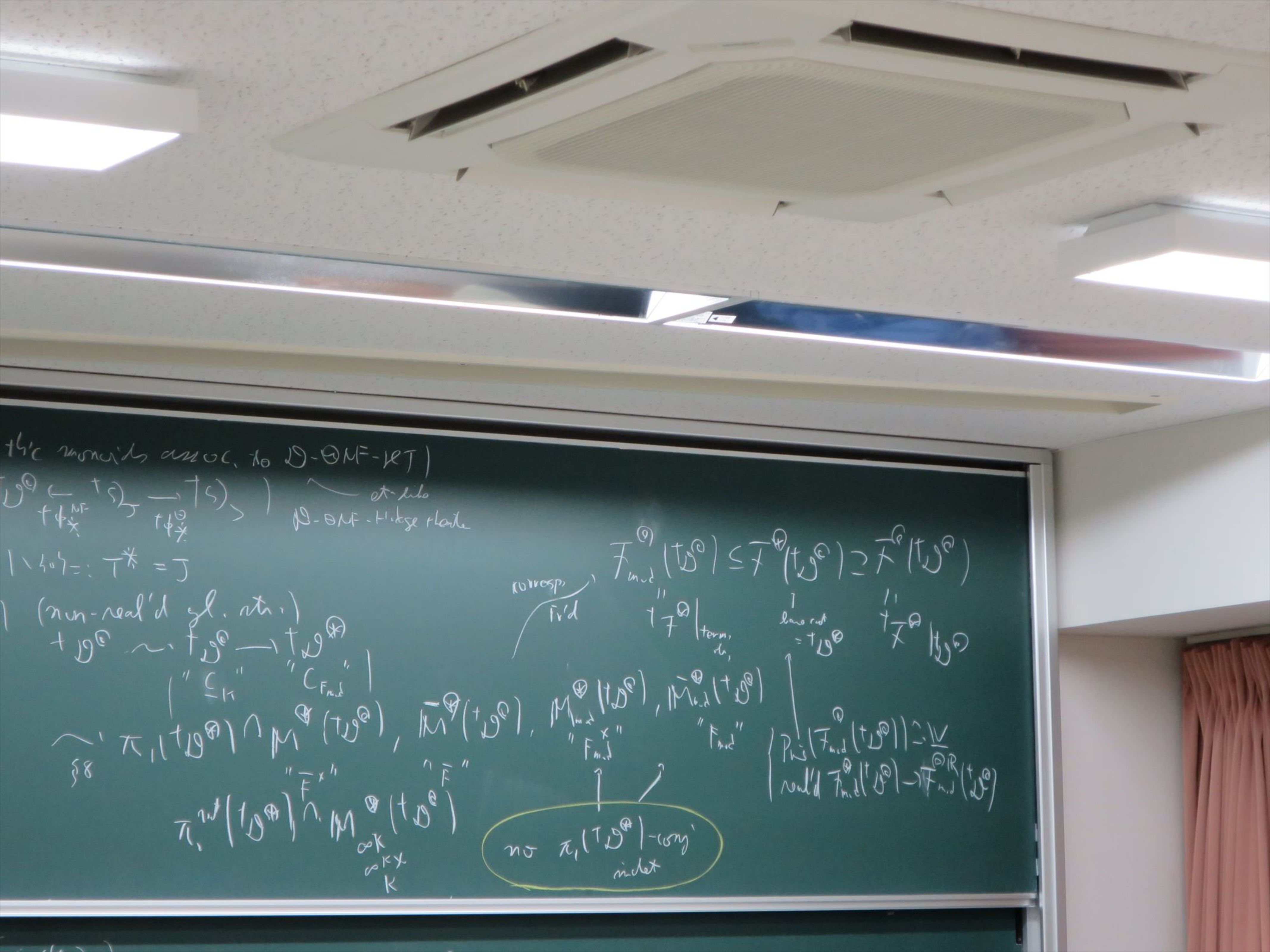
$(^\dagger\mathfrak{S})_\xi$ LabCusp

$^\dagger\phi^{\odot}_{\xi}$ LabCusp$(^\dagger\mathfrak{S})_>$

$^\dagger\mathfrak{S})$ LabCusp $(^\dagger\mathfrak{S})_>$

$\langle\ \rangle\ \tilde{F}\ \circ$ sgn $(\ldots,j^2,\ldots)$

ell. ~ for μ



th'c monoids assoc. to $\mathfrak{D}$-ΘNF-HT)

$\mathfrak{D}^{\circledcirc} \xleftarrow[{}^\dagger\phi^{NF}_*]{} {}^\dagger\mathcal{S}_J \xrightarrow[{}^\dagger\phi^{\Theta}_*]{} {}^\dagger\mathcal{S}_>$) ⌒ ét-like

$\mathfrak{D}$-ΘNF-Hodge theater

$| \setminus \{0\} =: T^* = J$

) (non-real'd gl. rtn.)

$^\dagger\mathfrak{D}^{\circledcirc} \rightsquigarrow {}^\dagger\mathfrak{D}^{\circledcirc} \longrightarrow {}^\dagger\mathfrak{D}^{\circledast}$

("$\underline{C}_K$" "$C_{F_{mod}}$")

⌒ $\pi_1({}^\dagger\mathfrak{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}({}^\dagger\mathfrak{D}^{\circledcirc})$, $\overline{\mathbb{M}}^{\circledast}({}^\dagger\mathfrak{D}^{\circledcirc})$, $\mathbb{M}^{\circledast}_{mod}({}^\dagger\mathfrak{D}^{\circledcirc})$, $\overline{\mathbb{M}}^{\circledast}_{mod}({}^\dagger\mathfrak{D}^{\circledcirc})$

§8 "$\overline{F}^\times$" "$\overline{F}$" "$F^\times_{mod}$" "F_{mod}"

$\pi_1^{rat}({}^\dagger\mathfrak{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}_{\infty\kappa}({}^\dagger\mathfrak{D}^{\circledcirc})$

$\infty\kappa$
$\infty\kappa\times$
κ

corresp.
fr'd

$\mathcal{F}^{\circledcirc}_{mod}({}^\dagger\mathfrak{D}^{\circledcirc}) \subseteq \mathcal{F}^{\circledast}({}^\dagger\mathfrak{D}^{\circledcirc}) \supseteq \overline{\mathcal{F}}^{\circledcirc}({}^\dagger\mathfrak{D}^{\circledcirc})$

$\| \; {}^\dagger\mathcal{F}^{\circledcirc}|_{\text{term. ob.}}$

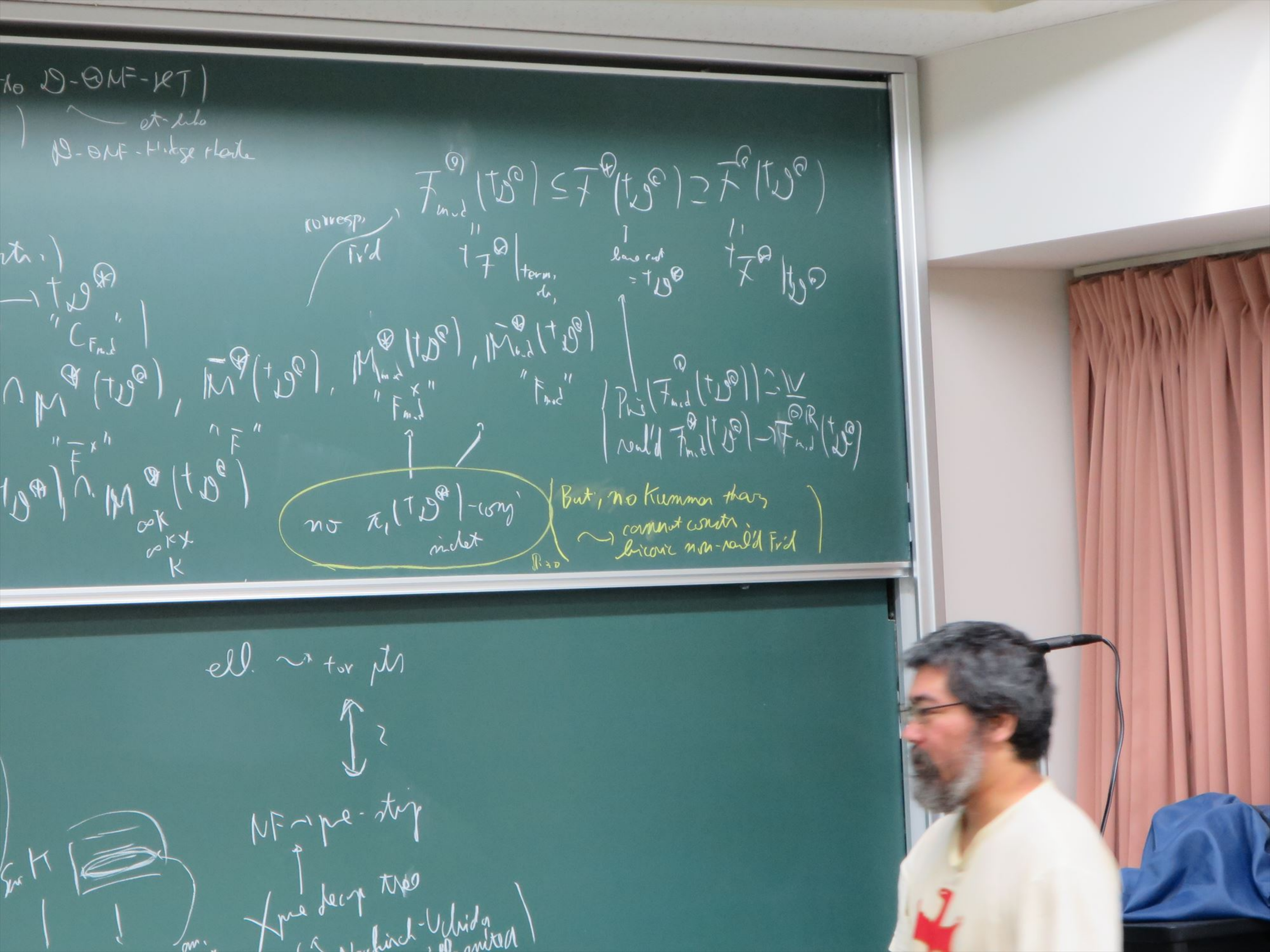
base cat. $= {}^\dagger\mathfrak{D}^{\circledast}$

$\| \; {}^\dagger\overline{\mathcal{F}}^{\circledcirc}|_{{}^\dagger\mathfrak{D}^{\circledcirc}}$

$\mathrm{Prime}(\mathcal{F}^{\circledcirc}_{mod}({}^\dagger\mathfrak{D}^{\circledcirc})) \xrightarrow{\sim} \underline{\mathbb{V}}$

real'd $\mathcal{F}^{\circledast}_{mod}({}^\dagger\mathfrak{D}^{\circledcirc}) \to \mathcal{F}^{\circledcirc\mathbb{R}}_{mod}({}^\dagger\mathfrak{D}^{\circledcirc})$

no $\pi_1({}^\dagger\mathfrak{D}^{\circledast})$-conj' indet


ϑ-ΘNF-HT)
et-like
ϑ-ΘNF-Hodge theater
corresp.
Fr'd
term.
base curve
But, no Kummer theory
cannot construct
bi-coric non-real'd Fr'd
no π₁(†D⊛)-conj'
indet
real'd
ell. ⇝ tor pts
NF-pre-step
Xpre decomp. theo
Nakamura-Uchida
Sur K

(ii) $j \in \mathrm{LabCusp}({}^\dagger\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} J \ (\cong \mathbb{F}_l^{*})$ $\quad (\underline{\mathbb{V}}^{\mathrm{Bor}} \twoheadrightarrow \underline{\mathbb{V}}^{\pm\mathrm{un}} \twoheadrightarrow \underline{\mathbb{V}}/l^2 \twoheadrightarrow \underline{\mathbb{V}}_{\mathrm{mod}},)$

$\overline{\mathcal{F}}^{\circledcirc}({}^\dagger\mathcal{D}^{\circledcirc}) \xrightarrow[\text{alg'm}]{\text{tuf}} \overline{\mathcal{F}}^{\circledcirc}({}^\dagger\mathcal{D}^{\circledcirc})|_j$ $\quad \mathcal{F}$-prime-strip well-def'd up to isom

$\mathbb{F}_l^{*} \curvearrowright {}^\dagger\mathcal{D}^{\circledcirc}$ $\quad \xrightarrow[\text{isom's}]{\text{induces}}$ $\left\{ \overline{\mathcal{F}}^{\circledcirc}({}^\dagger\mathcal{D}^{\circledcirc})|_j, \ \mathcal{M}^{(-)\circledast}_{\mathrm{mod}}({}^\dagger\mathcal{D}^{\circledcirc})_j, \right.$

poly-action $\quad$ [illegible] $\quad \overline{\mathcal{F}}^{\circledcirc}_{\mathrm{mod}}({}^\dagger\mathcal{D}^{\circledcirc})_j \to \overline{\mathcal{F}}^{\circledcirc\mathbb{R}}_{\mathrm{mod}}({}^\dagger\mathcal{D}^{\circledcirc})_j$

$\mathrm{Aut}_{\underline{\epsilon}}$ $\quad \left. \pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}^{\circledast}) \curvearrowright \mathcal{M}^{\circledast}_{\infty\kappa}({}^\dagger\mathcal{D}^{\circledast}) \right\}_j$

for distinct j's $\quad \underline{\mathbb{F}_l^{*}\text{-symmetrizing isom}}$

$(\cong \mathbb{F}_l^{\rtimes *})$

$$\underline{\mathbb{V}}^{\mathrm{Bor}} \hookrightarrow \underline{\mathbb{V}}^{\pm\mathrm{un}} \twoheadrightarrow \underline{\mathbb{V}}/\pm$$
$$\underline{\mathbb{V}}^{\pm\mathrm{un}} \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}},$$

$\mathcal{D}^{\circledcirc})|_{j}$, $\mathcal{F}$-prime-strip

well-def. up to isom

es

$\mathcal{F}^{\circledcirc}({}^\dagger\mathcal{D}^{\circledcirc})|_j$, ${}^{(-)}\mathcal{M}^{\circledast}_{\mathrm{mod}}({}^\dagger\mathcal{D}^{\circledcirc})_j$,

sm's

al.

$\mathcal{F}^{\circledcirc}_{\mathrm{mod}}({}^\dagger\mathcal{D}^{\circledcirc})_j \to \mathcal{F}^{\circledcirc\mathbb{R}}_{\mathrm{mod}}({}^\dagger\mathcal{D}^{\circledcirc})_j$,

$\{\pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}^{\circledast}) \curvearrowright \mathcal{M}^{\circledast}_{\infty\kappa}({}^\dagger\mathcal{D}^{\circledast})\}_j$,

for distinct j's

$\mathbb{F}_l^{\rtimes *}$-symmetrizing isom

$(-)_{\langle \mathbb{F}_l^{*}\rangle} \subset \prod_{j\in\mathbb{F}_l^{*}} (-)_j$

mod

others $\rightsquigarrow (\ )_{\langle \mathbb{F}_l^{*}\rangle}$

symbol

for $\mathbb{F}_l^{*}$-symm. isoms

(iii) $\mid$ lar

$j \in |\mathrm{LabCu}$

${}^\dagger\mathcal{D}^{\circledcirc}$,

$\prod_{j \in \mathbb{F}_l^*} (-)_j$

$n_l^* \rangle$

d

$*$. symm. isoms

(iii) (local factors & real'd gl. rtn.)

$j \in \mathrm{LabCusp}(^\dagger\mathcal{D}^{\circledcirc})$

$$^\dagger\mathcal{D}^{\circledcirc} \xrightarrow[\text{fn alg'm}]{}$$

Fr'd

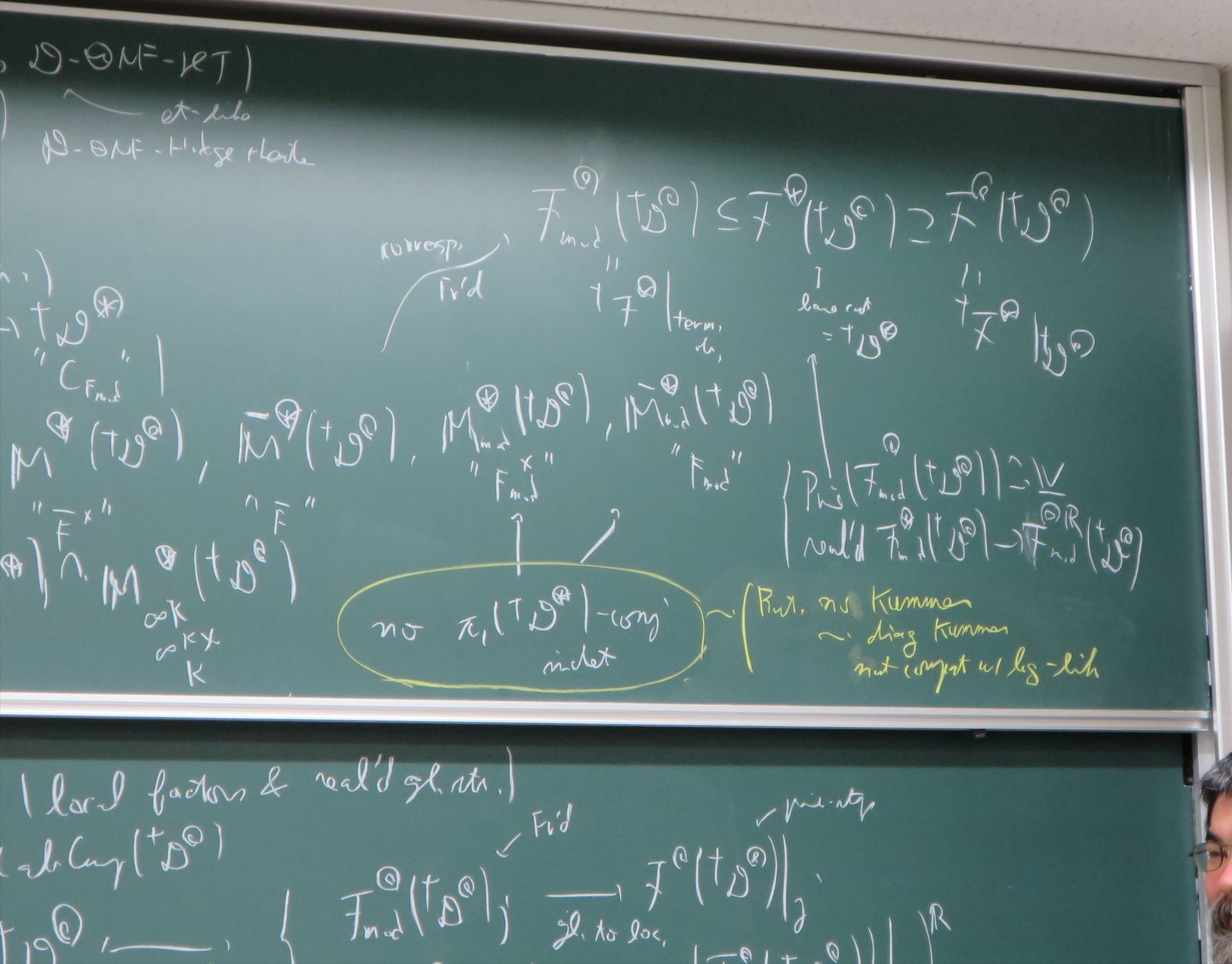
pri.-stp

$$\mathcal{F}_{\mathrm{mod}}^{\circledcirc}(^\dagger\mathcal{D}^{\circledcirc})_j \xrightarrow[\text{gl. to loc.}]{} \mathcal{F}^{\circledcirc}(^\dagger\mathcal{D}^{\circledcirc})|_j$$

$$\mathcal{F}_{\mathrm{mod}}^{\circledcirc\mathbb{R}}(^\dagger\mathcal{D}^{\circledcirc})_j \xrightarrow[\text{gl. to loc.}]{} (\mathcal{F}^{\circledcirc}(^\dagger\mathcal{D}^{\circledcirc})|_j)^{\mathbb{R}}$$

$$\{\pi_1^{\mathrm{rat}}(^\dagger\mathcal{D}^{\circledcirc}) \curvearrowright \mathbb{M}^{\circledcirc}_{\infty\kappa\times}(^\dagger\mathcal{D}^{\circledcirc}) \to \mathbb{M}_{\infty\kappa}(^\dagger\mathcal{D}_{\underline{v}_j}) \subseteq \mathbb{M}_{\infty\kappa\times\mu}(^\dagger\mathcal{D}_{\underline{v}_j})\}_{\underline{v} \in \underline{\mathbb{V}}}$$

$$\begin{tikzcd} \mathcal{D}^{\Vdash}\left((\mathcal{F}^{\circledcirc}(^\dagger\mathcal{D}^{\circledcirc})|_j)^{\mathcal{D}\vdash}\right) \arrow[r, "\sim"] & \mathcal{F}_{\mathrm{mod}}^{\circledcirc\mathbb{R}}(^\dagger\mathcal{D}^{\circledcirc})_j \\ \mathbb{R}_{\geq 0}\left(\mathcal{F}^{\circledcirc}(^\dagger\mathcal{D}^{\circledcirc})|_j\right)^{\mathcal{D}\vdash} \arrow[u, "\sim"] \arrow[r, "\text{assoc. } \mathcal{D}^{\vdash}\text{-pr-stp}", "\sim"'] & \mathcal{F}_{\mathcal{I}}(\mathcal{F}^{\circledcirc}(^\dagger\mathcal{D}^{\circledcirc})|_j)^{\mathbb{R}} \end{tikzcd}$$



$\mathcal{D}$-ΘNF-HT ← ét-like

ΘNF-HT ← Frd-like

↘

similar obj's + Kummer

([IUTchII, Cor4.8])

[IUTchII, Prp4.9]

(i) $G\ (\overset{?}{\cong} G_{\underline{v}}) \longmapsto G \curvearrowright \mathcal{O}^{\triangleright}(G), \mathcal{O}$

$\mathcal{C} \xrightarrow[\text{Frd}]{} \mathcal{D}$ $A \in^{\mu_0} \mathrm{Ob}($

unimen

X-Kummer str. on $\mathcal{C}$

X_{μ}-Kummer str. on $\mathcal{C}$

mer

n4.8])

[IUTch II, Prop 4.9]

(i) $G\ (\cong G_{\underline{k}})$

$\longmapsto G \curvearrowright \mathcal{O}^{\triangleright}(G), \mathcal{O}^{\times\mu}(G)$

$\mathcal{C} \xrightarrow[\text{Frd}]{} \mathcal{D}$ $A \in^{\mu_0} \mathrm{Ob}(\mathcal{D})$

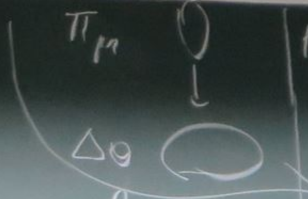
universal covering obj

internal

external

×-Kummer str. on $\mathcal{C}$ $\overset{\text{def}}{\Longleftrightarrow}$ a $\widehat{\mathbb{Z}}^{\times}$-orbit of isom $\mathcal{O}^{\times}(G) \xrightarrow{\sim} \mathcal{O}^{\times}(A)$

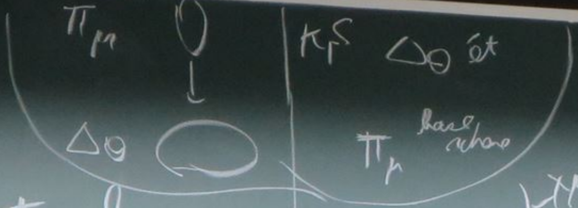
×μ-Kummer str. on $\mathcal{C}$ $\overset{\text{def}}{\Longleftrightarrow}$ an Ism-orbit of isom $\mathcal{O}^{\times\mu}(G) \xrightarrow{\sim} \mathcal{O}^{\times\mu}(A)$



$K_\Gamma^{\subseteq}$ Δ_Θ ét

Π_μ base schem

(split)



l

ering obj:

internal external

a $\widehat{\mathbb{Z}}^{\times}$-orbit of
isom $\mathcal{O}^{\times}(G) \xrightarrow{\sim} \mathcal{O}^{\times}(A)$

an Ismet-orbit of
isom $\mathcal{O}^{\times\mu}(G) \xrightarrow{\sim} \mathcal{O}^{\times\mu}(A)$

$\kappa^{\times\mu}$: $\times\mu$-Kummer str.

$H \underset{\text{open}}{\subset} G$

$\mapsto \mathcal{I}^{\kappa}_{H}(A) := \mathrm{Im}\{\mathcal{O}^{\times}(G)^{H} \to \mathcal{O}^{\times\mu}(G) \xrightarrow{\kappa^{\times\mu}} \mathcal{O}^{\times\mu}(A)\}$

(split) $\underline{\times\text{-Kummer Fr'd}}$ $\overset{\text{}}{\underset{\text{def}}{\Longleftrightarrow}}$ Fr'd w/ $\times$-Kummer str.
$\times\mu$-   (split) $\times\mu$-

(ii) $^{\dagger}\mathfrak{F}^{\vdash} = \{ ^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}} \}_{\underline{v} \in \underline{\mathbb{V}}}$ split Frd

$\}$ $\mathcal{F}^{\vdash}$-prime-strip

$^{\dagger}\mathfrak{D}^{\vdash} = \{ ^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}} \}_{\underline{v} \in \underline{\mathbb{V}}}$

$\underline{v}$: bad

$^{\dagger}A \in \mathrm{Mo\text{-}Ob}(^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}})$

univ. cov. obj.

$^{\dagger}G := \mathrm{Aut}(^{\dagger}A)$

$\mu_{2l}(^{\dagger}A) \subset \mathcal{O}^{\times}(^{\dagger}A) \subset \mathcal{O}^{\triangleright}(^{\dagger}A)$

$\mathcal{O}^{\triangleright}(^{\dagger}A) \supset \mathcal{O}^{\perp}(^{\dagger}A) := \langle \mu_{2l}(^{\dagger}A), \text{the image of splitting given by } ^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}} \rangle$

$\downarrow \quad \left(\mu_{2l} \cdot q^{\mathbb{N}}_{\underline{v}}, \ \mu_{2l} \cdot \underline{\underline{\Theta}}^{\mathbb{N}}_{\underline{v}}, \ \mu_{2l}\{\underline{\underline{q}}^{j^2}_{\underline{v}}\}^{l^*} \right)$

$\mathcal{O}^{\top}(^{\dagger}A) := \mathcal{O}^{\perp}(^{\dagger}A)/\mu_{2l}(^{\dagger}A) \overset{\sim}{\hookrightarrow} \mathcal{O}^{\triangleright}(^{\dagger}A)/\mathcal{O}^{\times}(A)$

$\mathcal{D}$-ΘNF-HT ← ét-like

[IUTch II, Prp 4.9]

(i)

$\mathcal{O}^{\triangleright\times\mu}({}^\dagger A) := \mathcal{O}^{\triangleright}({}^\dagger A) \times \mathcal{O}^{\times\mu}({}^\dagger A)$

${}^\dagger G \mapsto ({}^\dagger G \curvearrowright \mathcal{O}^\times({}^\dagger G))$

$\rightsquigarrow \exists!\ \hat{\mathbb{Z}}^\times$-orbit of isom

${}^\dagger\kappa_{\underline{v}}^{\vdash\times} : \mathcal{O}^\times({}^\dagger G) \xrightarrow{\sim} \mathcal{O}^\times({}^\dagger A)$

w/ ${}^\dagger G$-action.

$\Big\}$ modulo $\mathcal{O}^\mu(-)$

Ismet-orbit of isom

${}^\dagger\kappa_{\underline{v}}^{\vdash\times\mu} : \mathcal{O}^{\times\mu}({}^\dagger G) \xrightarrow{\sim} \mathcal{O}^{\times\mu}({}^\dagger A)$

by ${}^\dagger\mathcal{F}_{\underline{v}}^{\vdash}$)

$\{\underline{q}^{j^2}\}_{j=1}$)

${}^\dagger A)/\mathcal{O}^\times(A)$

${}^\dagger G \curvearrowright \mathcal{O}^{\triangleright\times\mu}({}^\dagger A)$

& divisor monoid

$\rightsquigarrow$ split Frd

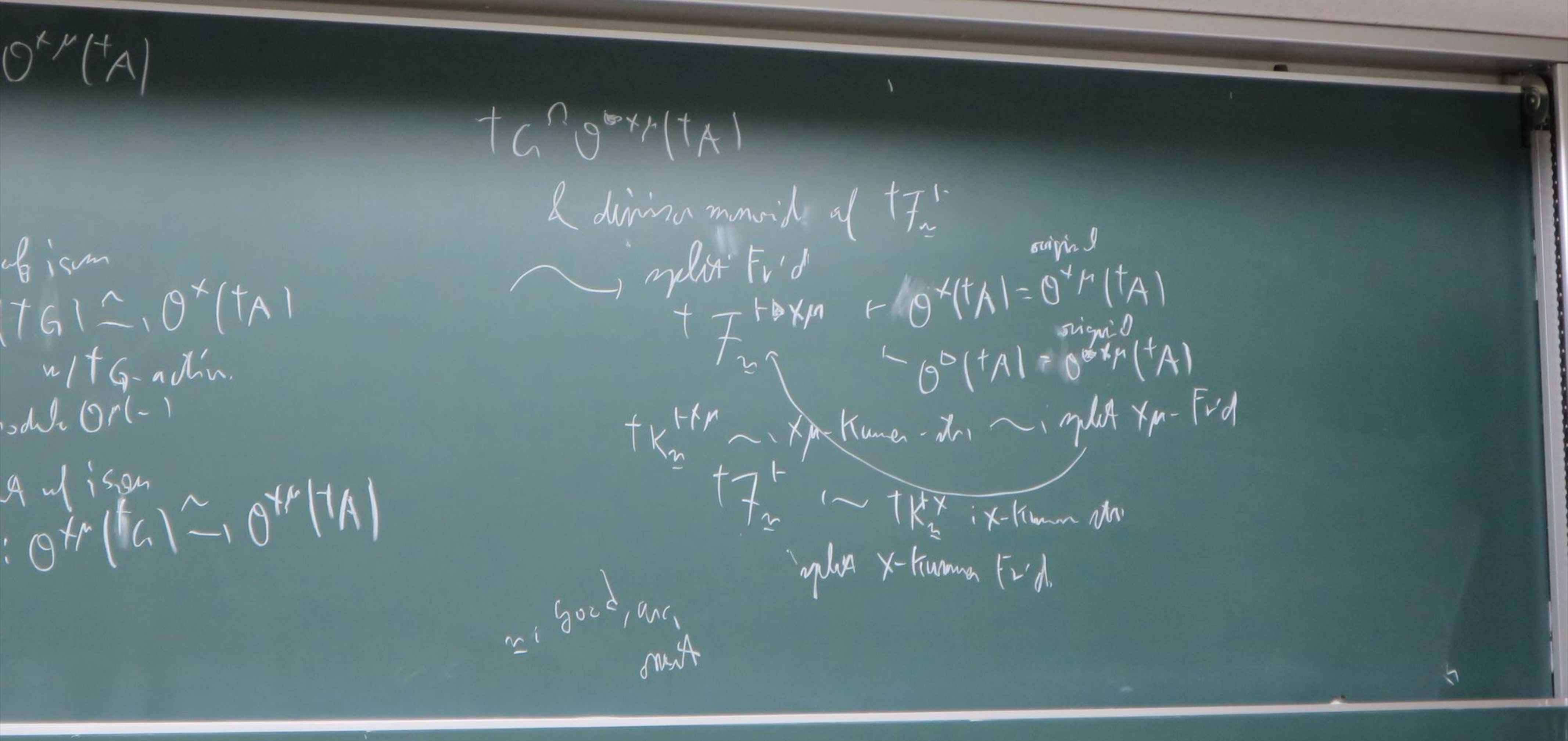
${}^\dagger\mathcal{F}_{\underline{v}}^{\vdash\triangleright\times\mu}$

${}^\dagger\kappa_{\underline{v}}^{\vdash\times\mu} \rightsquigarrow {}^{\times\mu}$

${}^\dagger\mathcal{F}_{\underline{v}}^{\vdash}$

spl

$\underline{v}$: good, arc, nonA



$\mathcal{O}^{\times\mu}({}^\dagger A)$

of isom

${}^\dagger G) \xrightarrow{\sim} \mathcal{O}^{\times}({}^\dagger A)$

w/ ${}^\dagger G$-action.

modulo $\mathcal{O}^{\mu}(-)$

of isom

: $\mathcal{O}^{\times\mu}({}^\dagger G) \xrightarrow{\sim} \mathcal{O}^{\times\mu}({}^\dagger A)$

${}^\dagger G \curvearrowright \mathcal{O}^{\triangleright\times\mu}({}^\dagger A)$

& divisor monoid of ${}^\dagger\mathcal{F}^{\vdash}_{\underline{\underline{v}}}$

$\rightsquigarrow$ split Fr'd

${}^\dagger\mathcal{F}^{\vdash\triangleright\times\mu}_{\underline{\underline{v}}} \leftarrow \mathcal{O}^{\times}({}^\dagger A) = \overset{\text{original}}{\mathcal{O}^{\times\mu}}({}^\dagger A)$

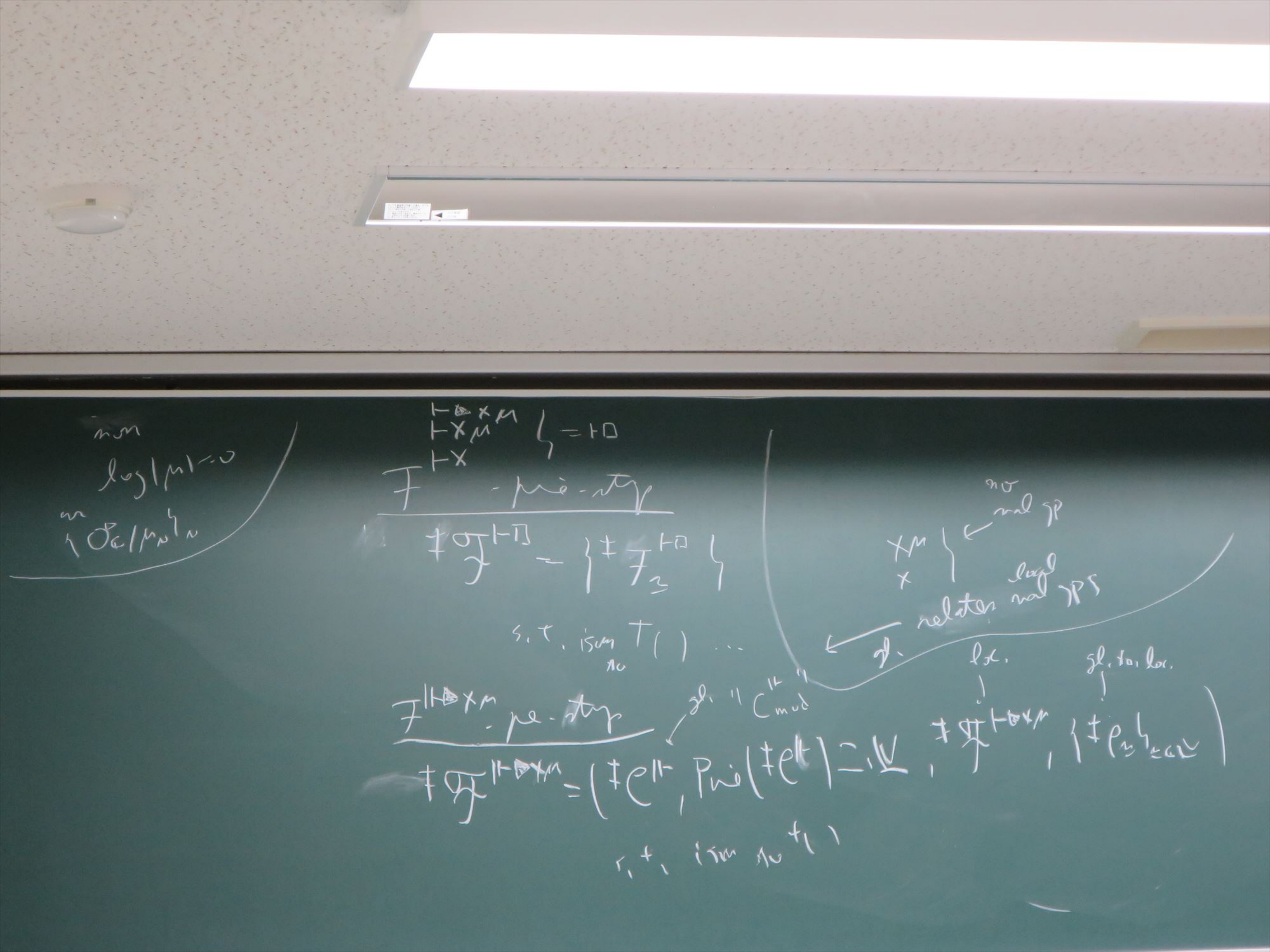
$\leftarrow \mathcal{O}^{\triangleright}({}^\dagger A) = \overset{\text{original}}{\mathcal{O}^{\triangleright\times\mu}}({}^\dagger A)$

${}^\dagger\mathcal{K}^{\vdash\times\mu}_{\underline{\underline{v}}} \rightsquigarrow$ $\times\mu$-Kummer-thr $\rightsquigarrow$ split $\times\mu$-Fr'd

${}^\dagger\mathcal{F}^{\vdash}_{\underline{\underline{v}}} \rightsquigarrow {}^\dagger\mathcal{K}^{\vdash\times}_{\underline{\underline{v}}}$: $\times$-Kummer thr

split $\times$-Kummer Fr'd.

$\underline{\underline{v}}$: good, arc, [illegible]



non
$\log|\mu| = 0$

non
$\{\mathcal{O}_{\bar{k}}/\mu_N\}_N$

$$\left.\begin{array}{l}\vdash\blacktriangle\times\mu\\ \vdash\times\mu^{\mu}\\ \vdash\times\end{array}\right\} = \vdash\Box$$

$\mathcal{F}^{\vdash\Box}$-prime-strip

$$^{\ddagger}\mathfrak{F}^{\vdash\Box} = \{^{\ddagger}\mathcal{F}^{\vdash\Box}_{\underline{v}}\}$$

s.t. $\underset{\text{nc}}{\text{isom}}$ $\dagger(\ )$ …

$\left.\begin{array}{l}\times\mu\\ \times\end{array}\right\} \leftarrow$ no real gp

local real gps
relates
$\leftarrow$ gl.

$\mathcal{F}^{\Vdash\blacktriangleright\times\mu}$-prime-strip

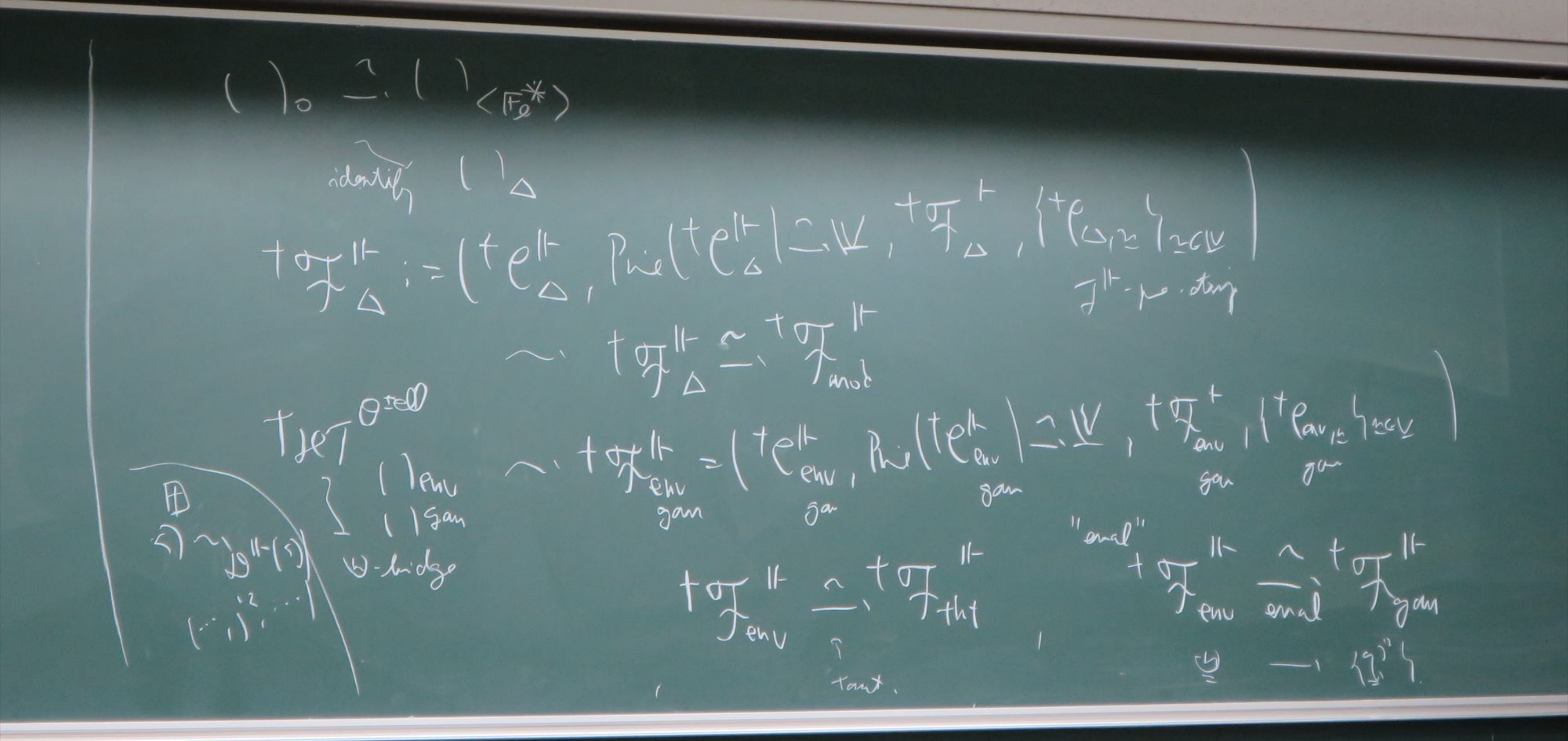
gl. "$\mathcal{C}^{\Vdash}_{\text{mod}}$"

loc.

gl. to loc.

$$^{\ddagger}\mathfrak{F}^{\Vdash\blacktriangleright\times\mu} = \left(^{\ddagger}\mathcal{C}^{\Vdash},\ \mathrm{Prime}(^{\ddagger}\mathcal{C}^{\Vdash}) \xrightarrow{\sim} \underline{\mathbb{V}},\ ^{\ddagger}\mathfrak{F}^{\vdash\blacktriangleright\times\mu},\ \{^{\ddagger}\rho_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}\right)$$

s.t. isom nc $\dagger(\ )$



$(\ )_{\circ} \overset{\sim}{\to} (\ )_{\langle \mathbb{F}_l^{*} \rangle}$

identify $(\ )_{\Delta}$

$${}^{\dagger}\mathfrak{F}^{\Vdash}_{\Delta} := \left({}^{\dagger}\mathcal{C}^{\Vdash}_{\Delta},\ \mathrm{Prime}({}^{\dagger}\mathcal{C}^{\Vdash}_{\Delta}) \overset{\sim}{\to} \underline{\mathbb{V}},\ {}^{\dagger}\mathfrak{F}^{\vdash}_{\Delta},\ \{{}^{\dagger}\rho_{\Delta,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}\right)$$

$\mathcal{D}^{\vdash}$-prime-strip

$$\rightsquigarrow\ {}^{\dagger}\mathfrak{F}^{\Vdash}_{\Delta} \overset{\sim}{\to} {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$$

${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}}$

$\left.\begin{matrix}(\ )_{\mathrm{env}}\\ (\ )_{\mathrm{gau}}\end{matrix}\right\}$

Θ-bridge

$$\rightsquigarrow\ {}^{\dagger}\mathfrak{F}^{\Vdash}_{\substack{\mathrm{env}\\ \mathrm{gau}}} = \left({}^{\dagger}\mathcal{C}^{\Vdash}_{\substack{\mathrm{env}\\ \mathrm{gau}}},\ \mathrm{Prime}({}^{\dagger}\mathcal{C}^{\Vdash}_{\substack{\mathrm{env}\\ \mathrm{gau}}}) \overset{\sim}{\to} \underline{\mathbb{V}},\ {}^{\dagger}\mathfrak{F}^{\vdash}_{\substack{\mathrm{env}\\ \mathrm{gau}}},\ \{{}^{\dagger}\rho_{\substack{\mathrm{env}\\ \mathrm{gau}},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}\right)$$

田

$\underline{q} \rightsquigarrow \underline{\underline{\Theta}}^{\Vdash}(\underline{q})$

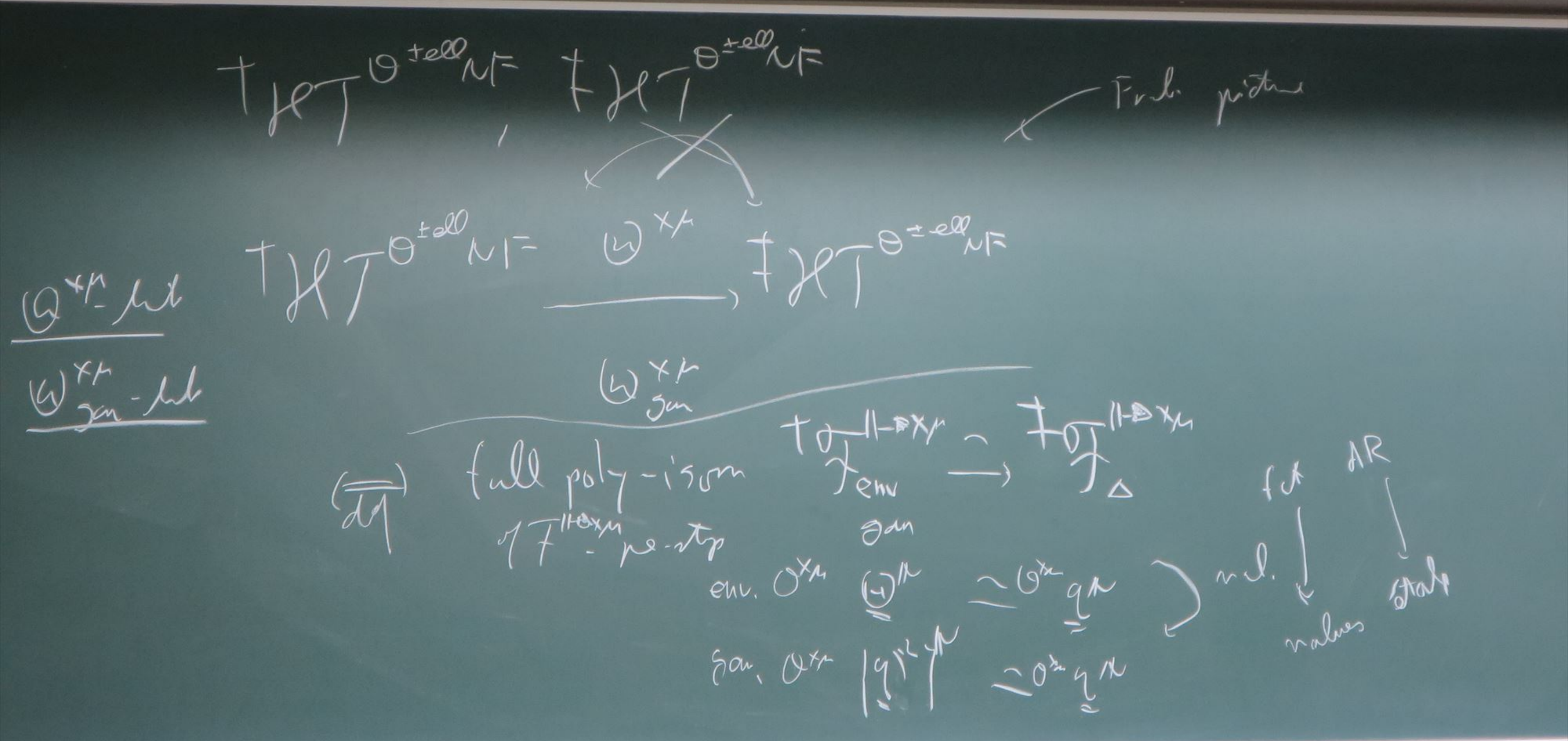
$(\ldots, \underline{q}^{j^2}, \ldots)$

$${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}} \underset{\substack{\uparrow\\ \mathrm{taut.}}}{\overset{\sim}{\to}} {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{tht}},$$

"eval"

$${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{env}} \underset{\mathrm{eval}}{\overset{\sim}{\to}} {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{gau}}$$

$$\underline{q} \mapsto \{\underline{q}^{j^2}\}$$



$^{\dagger}\mathcal{HT}^{\Theta^{\pm ell}NF}$, $^{\ddagger}\mathcal{HT}^{\Theta^{\pm ell}NF}$ ← Frob. picture

$$^{\dagger}\mathcal{HT}^{\Theta^{\pm ell}NF} \xrightarrow[\Theta^{\times\mu}_{gau}]{\Theta^{\times\mu}} {}^{\ddagger}\mathcal{HT}^{\Theta^{\pm ell}NF}$$

$\Theta^{\times\mu}$-link

$\Theta^{\times\mu}_{gau}$-link

$(\overline{\overline{dg}})$ full poly-isom $^{\dagger}\mathcal{F}^{\Vdash\blacktriangleright\times\mu}_{env} \xrightarrow{\sim} {}^{\ddagger}\mathcal{F}^{\Vdash\blacktriangleright\times\mu}_{\Delta}$

of $\mathcal{F}^{\Vdash\blacktriangleright\times\mu}$-pr-strip

gau

env. $\mathcal{O}^{\times\mu}$ $\underline{\Theta}^{\mu}$ $\simeq \mathcal{O}^{\times\mu}\underline{q}^{\mu}$ rel.

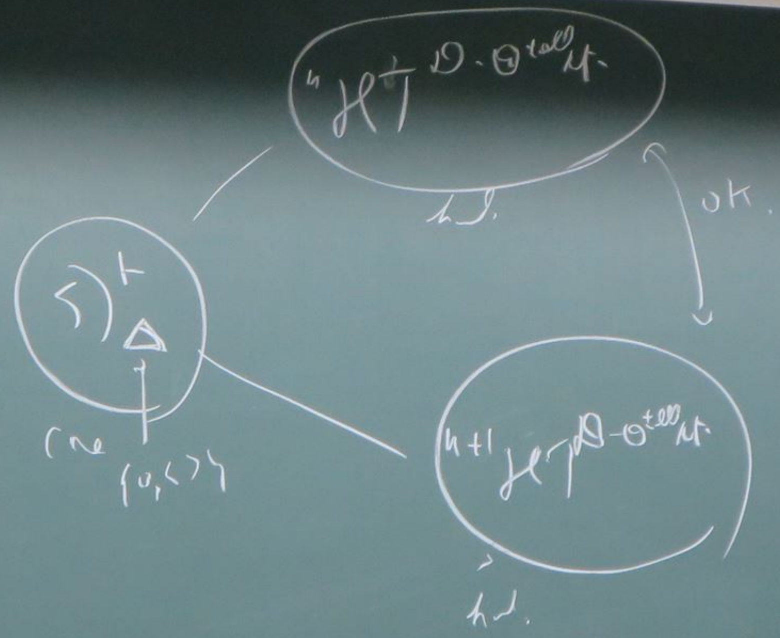
gau. $\mathcal{O}^{\times\mu}$ $\{\underline{q}^{j^2}\}^{\mu}$ $\simeq \mathcal{O}^{\times\mu}\underline{q}^{\mu}$

fct ↓ values

AR ↓ stab

picture

$$\begin{tikzcd}
\text{fct} \arrow[d, "\text{val.}"'] & \Delta R \arrow[d] \\
\text{values} & \text{étale}
\end{tikzcd}$$



§1, Log-links and Log-shells

[IUTch III, Def 1.1]

$(\ )^{\mathrm{gp}} = (\ )^{\mathrm{gp}} \cup \{0\}$

$^{\dagger}\mathfrak{F} = \{^{\dagger}\mathcal{F}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ $\mathcal{F}$-prime-strip $\longmapsto \Psi_{\mathrm{cns}}(^{\dagger}\mathfrak{F})$

$^{\dagger}\mathfrak{F}^{\vdash}, \ ^{\dagger}\mathfrak{F}^{\vdash\times\mu}$

$^{\dagger}\Pi_{\underline{v}}$

Belyi c

Arp. field

$\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$

($\underline{v}$: arc omit)

$$\text{“}\Psi_{^{\dagger}\mathcal{F}_{\underline{v}}} \supset \Psi^{\times}_{^{\dagger}\mathcal{F}_{\underline{v}}} \longrightarrow \overset{\sim}{\Psi}_{^{\dagger}\mathcal{F}_{\underline{v}}} := \left(\Psi^{\times}_{^{\dagger}\mathcal{F}_{\underline{v}}}\right)^{\mathrm{pf}} \quad \left(\supset \Psi_{\log}\right.$$

$$\text{“}\mathcal{O}^{\rhd}_{\overline{F}_{\underline{v}}} \supset \mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \longrightarrow \widehat{\overline{F}}_{\underline{v}} = \left(\mathcal{O}^{\times}_{\overline{F}_{\underline{v}}}\right)^{\mathrm{pf}} \quad \supset \mathcal{O}^{\rhd}_{\widetilde{F}}$$

$P \cup \{0\}$

Belyi cusp'zation

Arp. field str.

$:= \left(\mathcal{I}^{\times}_{{}^\dagger\mathfrak{F}_{\underline{v}}}\right)^{\mu\times} \supset \mathcal{I}_{\log({}^\dagger\mathfrak{F}_{\underline{v}})}$

$\parallel$

$= \left(\mathcal{O}^{\times}_{\bar{F}_v}\right)^{\mu\times} \supset \mathcal{O}^{\triangleright}_{\tilde{F}_v}$

$\{{}^\dagger\Pi_{\underline{v}} \curvearrowright \mathcal{I}_{\log({}^\dagger\mathfrak{F}_{\underline{v}})}\}$

$\rightsquigarrow$ Fr'd $\log({}^\dagger\mathfrak{F}_{\underline{v}})$

$\mathcal{I}^{\mathrm{gp}}_{\log({}^\dagger\mathfrak{F}_{\underline{v}})} \supseteq$

$\parallel$

$\left(\mathcal{O}^{\triangleright}_{\hat{F}_v}\right)^{\mathrm{gp}} \cup \{0\}$

diagram

$${}^\dagger\mathfrak{F}_{\underline{v}} \xrightarrow[\log]{} \log({}^\dagger\mathfrak{F}_{\underline{v}})$$

tautological log-link

${}^\dagger\mathfrak{F}$

l

$)\}$

$\mathcal{F}_{\underline{v}})$

diagram

$$^{\dagger}\mathcal{F}_{\underline{v}} \xrightarrow[\log]{} \log(^{\dagger}\mathcal{F}_{\underline{v}})$$

<u>tautological log-link</u>

$$^{\dagger}\mathfrak{F} = \{^{\dagger}\mathcal{F}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

$$\log(^{\dagger}\mathcal{F}_{\underline{v}}) \xrightarrow[(\text{poly})]{\sim} {}^{\ddagger}\mathcal{F}_{\underline{v}}$$

$$\wr$$

$$^{\dagger}\mathcal{F}_{\underline{v}} \xrightarrow{\log} {}^{\ddagger}\mathcal{F}_{\underline{v}}$$

<u>log-link</u>

full poly ⇝ <u>full log-link</u>

$$\widetilde{\mathcal{I}}_{^{\dagger}\mathcal{F}_{\underline{v}}} := \frac{1}{2p_{\underline{v}}} \mathrm{Im}\{(\overline{\Psi}^{\times}_{^{\dagger}\mathcal{F}_{\underline{v}}})^{G_{\underline{v}}(^{\dagger}\Pi_{\underline{v}})} \to \overline{\Psi}^{\sim}_{^{\dagger}\mathcal{F}_{\underline{v}}}\}$$

$$\cap$$

<u>log-shell</u>

$$\overline{\Psi}^{\sim}_{^{\dagger}\mathcal{F}_{\underline{v}}} = \overline{\Psi}^{\mathrm{gp}}_{\log(^{\dagger}\mathcal{F}_{\underline{v}})}$$

$$\underline{\log}({}^\dagger\mathfrak{F}) := \{ \underline{\log}({}^\dagger\mathcal{F}_{\underline{v}}) := \Psi^{\sim}_{{}^\dagger\mathcal{F}_{\underline{v}}} \}_{\underline{v}\in\underline{\mathbb{V}}}$$

$G_{\underline{v}}({}^\dagger\Pi_{\underline{v}})$ if $\underline{v}$: non-arch.

$$ {}^\dagger\mathfrak{F} \xrightarrow{\log} \log({}^\dagger\mathfrak{F}) \rightsquigarrow \{ {}^\dagger\mathcal{F}_{\underline{v}} \xrightarrow{\log} \log({}^\dagger\mathcal{F}_{\underline{v}}) \}_{\underline{v}\in\underline{\mathbb{V}}}$$

tautol. log-link

$$\log({}^\dagger\mathfrak{F}) \underset{(\mathrm{poly})}{\xrightarrow{\sim}} {}^\ddagger\mathfrak{F})$$

$$\rightarrow {}^\dagger\mathfrak{F} \xrightarrow{\log} {}^\ddagger\mathfrak{F}$$ log-link

full

<u>full log-link</u>

$\mathcal{I}$

$G_{\underline{v}}({}^\dagger\Pi_{\underline{v}})$

§1, Log-links and Log-Shells

[IUTch III, Def 1.1]

$(\ )^{\underline{\mathrm{gp}}} = (\ )^{\mathrm{gp}} \cup \{0\}$

$\}_{\underline{v}\in\underline{\mathbb{V}}}$

-ancl.

${}^{\dagger}\mathcal{F}_{\underline{v}} \xrightarrow{\log} \log({}^{\dagger}\mathcal{F}_{\underline{v}})\}_{\underline{v}\in\underline{\mathbb{V}}}$

log-lk

full log-lk

$$\begin{tikzcd}
G_{\underline{v}}({}^{\dagger}\Pi_{\underline{v}}) \curvearrowright & \mathcal{I}_{{}^{\dagger}\mathcal{F}_{\underline{v}}} \arrow[d, equal] & \subseteq & \underline{\log}({}^{\dagger}\mathcal{F}_{\underline{v}}) \arrow[d, equal] \\
& \ddot{\mathcal{I}}_{{}^{\dagger}\mathcal{F}_{\underline{v}}^{\vdash\times\mu}} & \subseteq & \underline{\ddot{\log}}({}^{\dagger}\mathcal{F}_{\underline{v}}^{\vdash\times\mu})
\end{tikzcd}$$

$$\mathcal{I}_{{}^{\dagger}\mathfrak{F}^{(\vdash\times\mu)}} := \{\mathcal{I}_{{}^{\dagger}\mathcal{F}_{\underline{v}}^{(\vdash\times\mu)}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

$$\underline{\log}({}^{\dagger}\mathfrak{F}^{(\vdash\times\mu)}) := \{\underline{\log}({}^{\dagger}\mathfrak{F}^{(\vdash\times\mu)}_{\underline{v}})\}_{\underline{v}\in\underline{\mathbb{V}}}$$

${}^{\dagger}\mathfrak{F} \xrightarrow{\log} {}^{\dagger}\mathfrak{F}$

$\sim, t_s) \xrightarrow[\text{(poly)}]{\sim}$

A

$\overline{\mathcal{I}}_{\text{cns}}({}^{\dagger}\mathfrak{F}$

final

not

SPL}...

$^{\vdash \times \mu})$

$\cong$

$^{\dagger}\mathcal{F}^{\vdash \times \mu}\}_{\cong \text{C.U}}$

$\{\underline{\log}(^{\dagger}\mathfrak{F}^{\vdash \times \mu})\}_{\cong \text{C.U}}$

$^{\dagger}\mathfrak{F} \xrightarrow[\sim]{\log} {}^{\ddagger}\mathfrak{F}$

$\rightsquigarrow {}^{\dagger}\mathcal{S}) \xrightarrow[(\text{poly})]{\sim} {}^{\ddagger}\mathcal{S}) \rightsquigarrow {}^{\dagger}\mathcal{S})^{\vdash} \xrightarrow[(\text{poly})]{\sim} {}^{\ddagger}\mathcal{S})^{\vdash}$

$A \overset{\text{bij}}{\xrightarrow{\sim}} \log(A)$

$\rightsquigarrow \mu^{\log}(A) = \mu^{\log}(\log(A))$

($\rightsquigarrow$ log-Kummer compat w/ log-vol.)

$\overline{\mathcal{F}}_{\text{cns}}(^{\dagger}\mathfrak{F}) \xrightarrow[\text{Kummer}]{\sim} \overline{\mathcal{F}}^{\text{cns}}_{\text{ét}}(^{\dagger}\mathcal{C})$

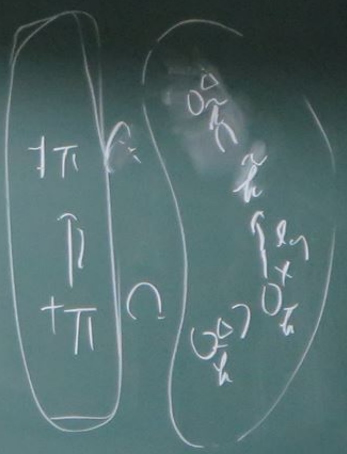
Frob.

~~~~ **not** compat w/ log-link

later replaced by ($\rightsquigarrow$ upper semi-compat.

$\mathcal{O}^{\times} \subset \frac{1}{2p}\log(\mathcal{O}^{\times})$

$\log(\mathcal{O}^{\times}) \subset$)



$\log$ "$a^{1/N}$" " $(\log a)^{1/N}$ "

$^{\dagger}\mathfrak{F} - \{^{\ddagger}\mathfrak{F}\}$
~~~~

$${}^{\dagger}\mathfrak{D}^{\vdash} = \{ {}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}} \}$$

$$\wr$$

$$\underline{\log}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}) := \{ {}^{\dagger}G_{\underline{v}} \curvearrowright k^{\wedge}({}^{\dagger}G_{\underline{v}}) \}$$

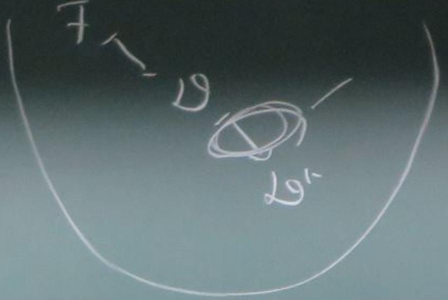
$$\cup$$

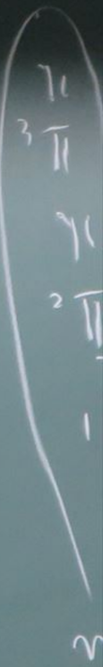
$$\tilde{\mathcal{I}}_{{}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}} := \tilde{\mathcal{I}}({}^{\dagger}G_{\underline{v}}) \qquad \frac{\text{mono-an.}}{\underline{\text{log-shell}}}$$

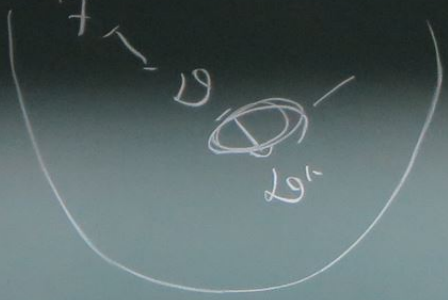
$${}^{\ddagger}\mathfrak{F}^{\vdash\times\mu} = \{ {}^{\ddagger}\mathcal{F}^{\vdash\times\mu}_{\underline{v}} \} \rightsquigarrow \underline{\log}({}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}) \xrightarrow[\sim]{\text{poly}} \underline{\log}({}^{\ddagger}\mathcal{F}^{\vdash\times\mu}_{\underline{v}})$$

Isomet-orbit

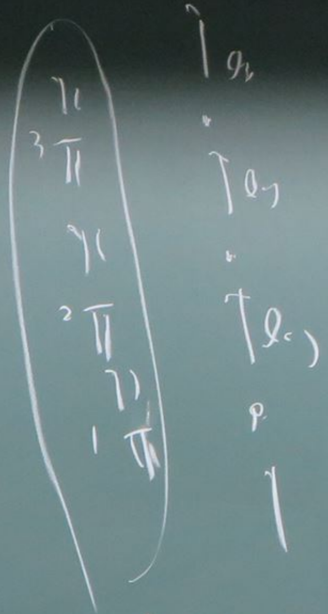
$$\left(\text{an } \{\pm 1\} \times \{\pm 1\} \right)$$







$\mathcal{F}_{\triangle}^{\vdash\times\mu})$



$^{n-1}\mathfrak{S}) \xrightarrow{\sim} {}^{n}\mathfrak{S}) \xrightarrow{\sim} {}^{n+1}\mathfrak{S})$

$^{\circ}\mathfrak{S})$

$\wr$

$\mathcal{I}_{{}^{\circ}\mathfrak{S})}(\mathcal{F}^{\Psi}_{\mathrm{cns}}({}^{\circ}\mathfrak{S}))$

(vert) conic hull, log-shell

vert. cor Frob. picture

[IUTchIII, Prp

Thm$^{\Theta}$

fix an in

$S) \xrightarrow{\sim} {}^{n+1}S)$

$)$

$S) \subset \overline{\mathcal{I}}^{\mathbb{Q}}_{\mathrm{cnc}}({}^{\circ}S))$

onic hull, log-shell

[IUTch III, Prop 1.3]

${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$, ${}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$

fix an isom $\Xi : {}^{\dagger}\mathcal{HT}^{D\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\sim} {}^{\ddagger}\mathcal{HT}^{D\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$

${}^{\dagger}\mathfrak{F}_{\square} = {}^{\dagger}\mathfrak{F}_{>}$, or ${}^{\dagger}\mathfrak{F}_{\succ}$, or ${}^{\dagger}\mathfrak{F}_{\vdash}$, or ${}^{\dagger}\mathfrak{F}_{\top}$

$\Xi \rightsquigarrow \Xi({}^{\dagger}\mathfrak{F}_{\square}) \xrightarrow{\sim} \Xi({}^{\ddagger}\mathfrak{F}_{\square})$

$\exists!\ \log({}^{\dagger}\mathfrak{F}_{\square}) \xrightarrow{\sim} {}^{\ddagger}\mathfrak{F}_{\square}$ $\hookleftarrow \left(\mathrm{Isom}({}^{\dagger}\mathfrak{F}, {}^{\ddagger}\mathfrak{F}) \xrightarrow{\sim} \mathrm{Isom}({}^{\dagger}S, {}^{\ddagger}S) \right)$

$\wr$

${}^{\dagger}\mathfrak{F}_{\square} \xrightarrow[\sim]{\log} {}^{\ddagger}\mathfrak{F}_{\square}$

$${}^{\dagger}\mathfrak{F}^{\Vdash} = \{{}^{\dagger}\mathcal{F}_{v}^{\Vdash\times\mu}\} \rightsquigarrow \underline{\log}({}^{\dagger}\mathcal{D}^{\vdash}_{\succ}) \overset{\text{poly}}{\xrightarrow{\sim}} \underline{\log}({}^{\dagger}\mathcal{F}_{\succ}^{\Vdash\times\mu})$$

Isomet-orbit

$({}^{\text{an}}\{\pm1\}\times\{\pm1\})$

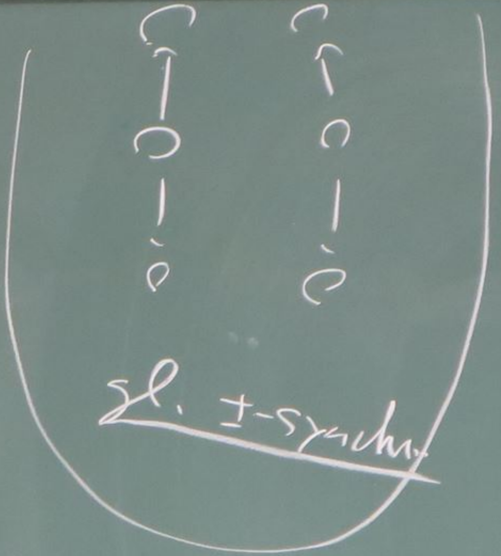
$\mathfrak{D} = \{\succ, \gamma, \mathcal{J}, \tau\}$

$\rightsquigarrow$ collection

$${}^{\dagger}\mathcal{H}\mathcal{T}^{\Theta^{\pm ell}NF} \xrightarrow{\log} {}^{\ddagger}\mathcal{H}\mathcal{T}^{\Theta^{\pm ell}NF}$$

$\subseteq \rightsquigarrow$ poly, full poly

<u>log-link</u> <u>full log-link</u>



$${}^{n+1}\mathcal{H}\mathcal{T}^{\Theta^{\pm ell}NF} \xrightarrow{\log} {}^{n}$$

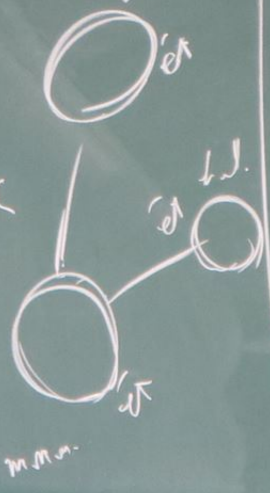
F

$\mathcal{L}og(\mathcal{F}_{\mathrm{LGP}}(\;))$

vert. co Frob.-picture (vert.) coric holo. log-shell

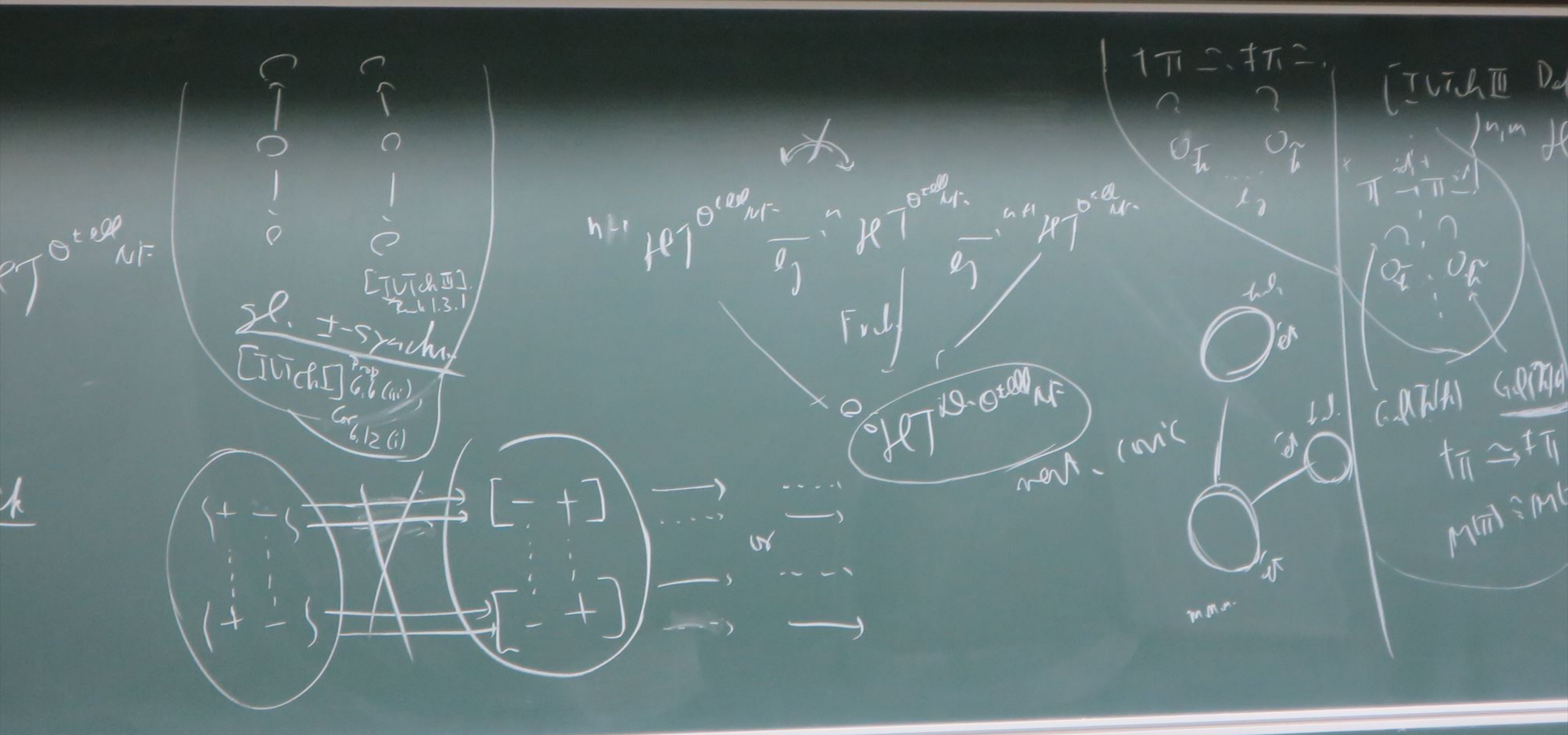
$$\begin{tikzcd}
{}^{n-1}\mathcal{HT}^{\Theta^{\pm ell}NF} \arrow[r, "\log"] \arrow[rrd, bend left=10] \arrow[dr] & {}^{n}\mathcal{HT}^{\Theta^{\pm ell}NF} \arrow[r, "\log"] \arrow[d, "\mathrm{Frob.}"'] & {}^{n+1}\mathcal{HT}^{\Theta^{\pm ell}NF} \arrow[dl] \\
& {}^{\circ}\mathcal{HT}^{D\text{-}\Theta^{\pm ell}NF} &
\end{tikzcd}$$

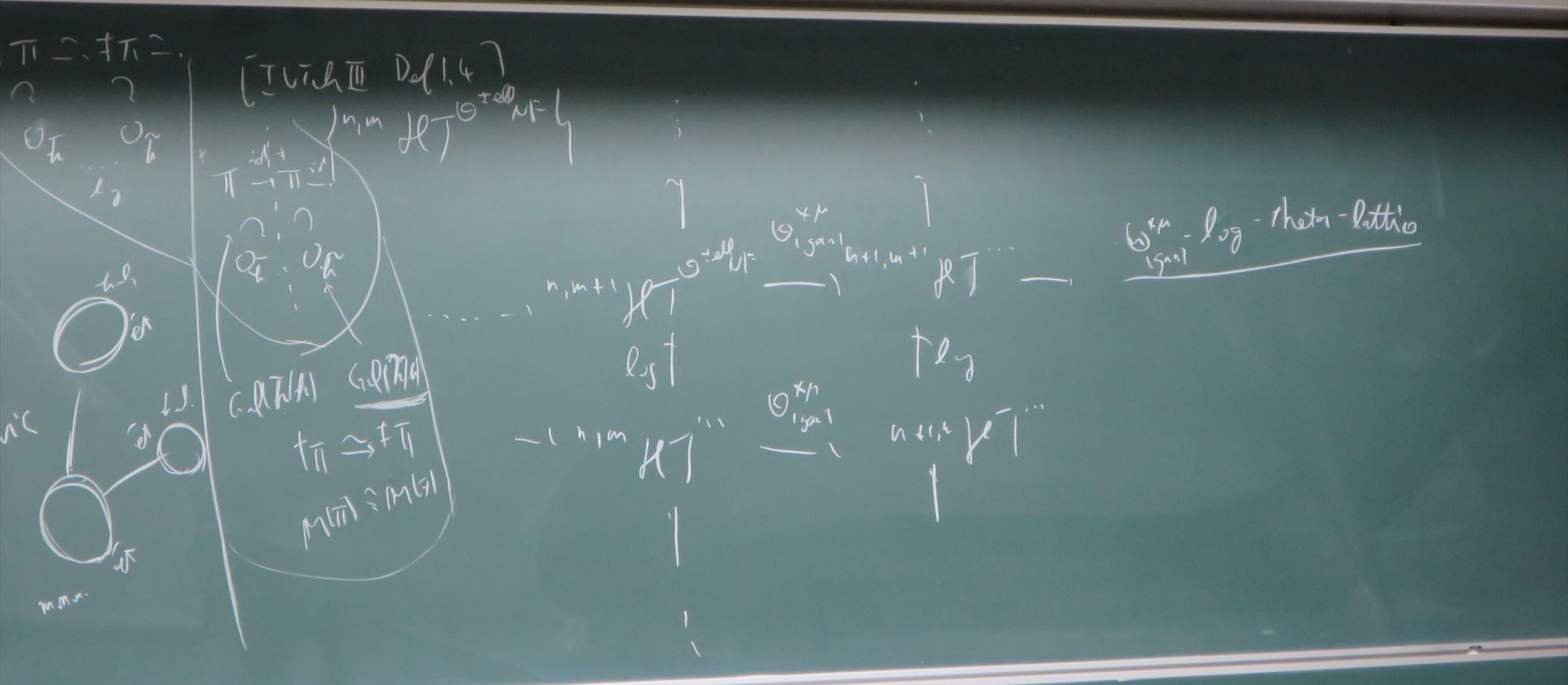
vert. coric



[IUTchIII Def 1.4]

$\{{}^{n,m}\mathcal{HT}^{\Theta^{\pm ell}NF}\}$


HT^Θ±ell NF
[IUTchIII] Rmk 1.3.1
cf. ±-synchro.
[IUTchI] Prop 6.6 (iii)
Cor 6.12 (i)
or
n-1 HT^Θ±ell NF
l_J
n HT^Θ±ell NF
n+1 HT^Θ±ell NF
Frob.
HT^D-Θ±ell NF
mono-conic
†Π ⊇ ‡Π ⊇
O_k̄ O_k̄
l_J
tht
et
mono
[IUTchIII Def
n,m
Gal(F̄/F) Gal(F̄/F)
†Π ⥲ ‡Π
M(Π) ≅ M(



[IUTch III Def 1.4]
$^{n,m}\mathcal{HT}^{\Theta^{\pm ell}NF}$
$\Pi \xrightarrow{id} \Pi$
$O_{\tilde{h}}$
$\log$
Gal(F̄/F)
Gal(F̄/F)
$^\dagger\Pi \xrightarrow{\sim} {}^\ddagger\Pi$
$M(\Pi) \cong M(\Pi)$
$^{n,m+1}\mathcal{HT}^{\Theta^{\pm ell}NF}$
$\Theta^{\times\mu}_{gau}$
$^{n+1,m+1}\mathcal{HT}$
$^{n,m}\mathcal{HT}$
$^{n+1,m}\mathcal{HT}$
$\Theta^{\times\mu}_{gau}$-log-theta-lattice

[IUTch III, Th 1.5] (bicoric of $\log$-theta-lattice) $\overset{\to\ \uparrow}{n,m}$

(i) (vert. coricity)

$$\xrightarrow[\text{induces}]{} \quad \xrightarrow[\text{full poly}]{\sim} \; {}^{n,m}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm ell}NF} \xrightarrow[\text{full poly}]{\sim} {}^{n,m+1}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm ell}NF} \xrightarrow[\text{full poly}]{\sim}$$

$$\left(\text{"}\, {}^{n,m}\Pi \xrightarrow{\sim} {}^{n,m+1}\Pi \xrightarrow{\sim} \cdots \text{"} \right)$$

(ii) (horizont. coricity)

$$\xrightarrow[\text{induces}]{} \quad \xrightarrow[\text{full poly}]{} \; {}^{n,m}\mathcal{F}^{\vdash\times\mu}_{\Delta} \xrightarrow[\text{full poly}]{\sim} {}^{n+1,m}\mathcal{F}^{\vdash\times\mu}_{\Delta} \xrightarrow{\sim} \cdots$$

$$\Delta = \{0, \langle \mathbb{F}_l^* \rangle\} \quad \text{"}\left({}^{n,m}G_{\underline{v}} \curvearrowright {}^{n,m}\mathcal{O}^{\times\mu}_{\underline{v}}\right) \xrightarrow{\sim} \left({}^{n+1,m}G_{\underline{v}} \curvearrowright {}^{n+1,m}\mathcal{O}^{\times\mu}_{\underline{v}}\right)\text{"}$$

${}^{\dagger}\Pi \xrightarrow{\sim} {}^{\ddagger}\Pi$ $\quad \mathcal{O}^{\triangleright} \subset \hat{\mathcal{O}}$ not full

poly-isom of (i), (ii)

$$\xrightarrow{\sim} \quad \mathcal{F}^{\vdash\times\mu}({}^{n,m}\mathfrak{S}) \,|\, \xrightarrow{\sim} \mathcal{F}^{\vdash\times\mu}({}^{n,m+1}\mathfrak{S}) \,|\, \xrightarrow[\text{poly}]{\sim} \cdots$$

$\sim {}^{n,m}\mathcal{F}_{\triangle} \xrightarrow{\sim} \mathcal{F}_{\triangle}(\ \mathfrak{D})_{\triangle})$

$\uparrow$ Frd. σ^{t}

compat'le w/ {×μ-Kummer str., poly isom. of (ii)}

$\Theta^{\times\mu}$)

(iv) (bi-coric mono-an. log-shells)
full poly of (iii) $\rightsquigarrow$ $\{\mathcal{I}_{{}^{n,m}\mathfrak{D}^{\vdash}_{\triangle}} \subseteq \underline{\log}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle})\} \xrightarrow[\sim]{\text{full poly}} \{\mathcal{I}_{{}^{n',m'}\mathfrak{D}^{\vdash}_{\triangle}} \subseteq \underline{\log}({}^{n',m'}\mathfrak{D}^{\vdash}_{\triangle})\}$

(iii) (bi-coric $\mathcal{F}^{\vdash\times\mu}$-prime-strips)

${}^{n,m}\mathfrak{D}^{\vdash}_{\triangle}$: $\mathcal{D}^{\vdash}$-prime-strip

${}^{n,m}\mathfrak{D}_{\succ}$: $\mathcal{D}$-prime-strip

⊞ $\rightarrow$ $\rightsquigarrow \overline{\Psi}_{\mathrm{cns}}({}^{n,m}\mathfrak{D}_{\succ})|_{0} \xrightarrow{\sim} \Psi_{\mathrm{cns}}({}^{n,m}\mathfrak{D}_{\succ})|_{\langle\mathbb{F}^{*}_{\ell}\rangle}$

$\mathcal{F}^{\vdash}_{\triangle}({}^{n,m}\mathfrak{D}_{\succ})\ (\xrightarrow{\sim} \mathcal{F}^{\vdash}_{\succ}({}^{n,m}\mathfrak{D}_{\succ}))$

$\supseteq \Psi_{\mathrm{cns}}({}^{n,m}\mathfrak{D}_{\succ})$

identify & def'd Frd

$\wr\ \times\mu$

$\mathcal{F}^{\vdash\times}_{\triangle}({}^{n,m}\mathfrak{D}_{\succ})$

$\overline{\Psi}_{\mathrm{cns}}({}^{n,m}\mathfrak{D}_{\succ})^{\times}_{\underline{v}} \xrightarrow{\sim} \overline{\Psi}^{\mathrm{ss}}_{\mathrm{cns}}({}^{n,m}\mathfrak{D}^{\vdash})^{\times}_{\underline{v}}$

${}^{n,m}\mathfrak{D}^{\vdash}_{\underline{v}}$

$\uparrow$ fd alg'm

$\mathcal{F}^{\vdash\times}_{\triangle}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle})$

w/ $\mathcal{F}^{\vdash\times}_{\triangle}({}^{n,m}\mathfrak{D}_{\succ}) \xrightarrow{\sim} \mathcal{F}^{\vdash\times}_{\triangle}({}^{n,m}\mathfrak{D}^{\vdash}_{\triangle})$

${}^{\dagger}\Pi \cong {}^{\ddagger}\Pi \qquad \mathcal{O}^{\circ} \subseteq \hat{\mathcal{O}}^{\times}$ not full

Kummer

poly-isom of (i), (ii)

$\rightsquigarrow$

$$\longrightarrow \mathcal{F}^{\vdash\times\mu}_{\triangle}({}^{n,m}\mathfrak{S}_{\succ})| \xrightarrow[\text{poly}]{\sim} \mathcal{F}^{\vdash\times\mu}_{\triangle}({}^{n,m+1}\mathfrak{S}_{\succ})| \xrightarrow[\text{poly}]{\sim} \cdots$$

$\rightsquigarrow$ ^{n,m}

$$\longrightarrow \mathcal{F}^{\vdash\times\mu}_{\triangle}({}^{n,m}\mathfrak{S}^{\vdash}_{\triangle})| \xrightarrow[\text{full poly}]{\sim} \mathcal{F}^{\vdash\times\mu}_{\triangle}({}^{n+1,m}\mathfrak{S}^{\vdash}_{\triangle})| \xrightarrow[\text{full poly}]{\sim} \cdots$$

${}^{\dagger}\Pi$

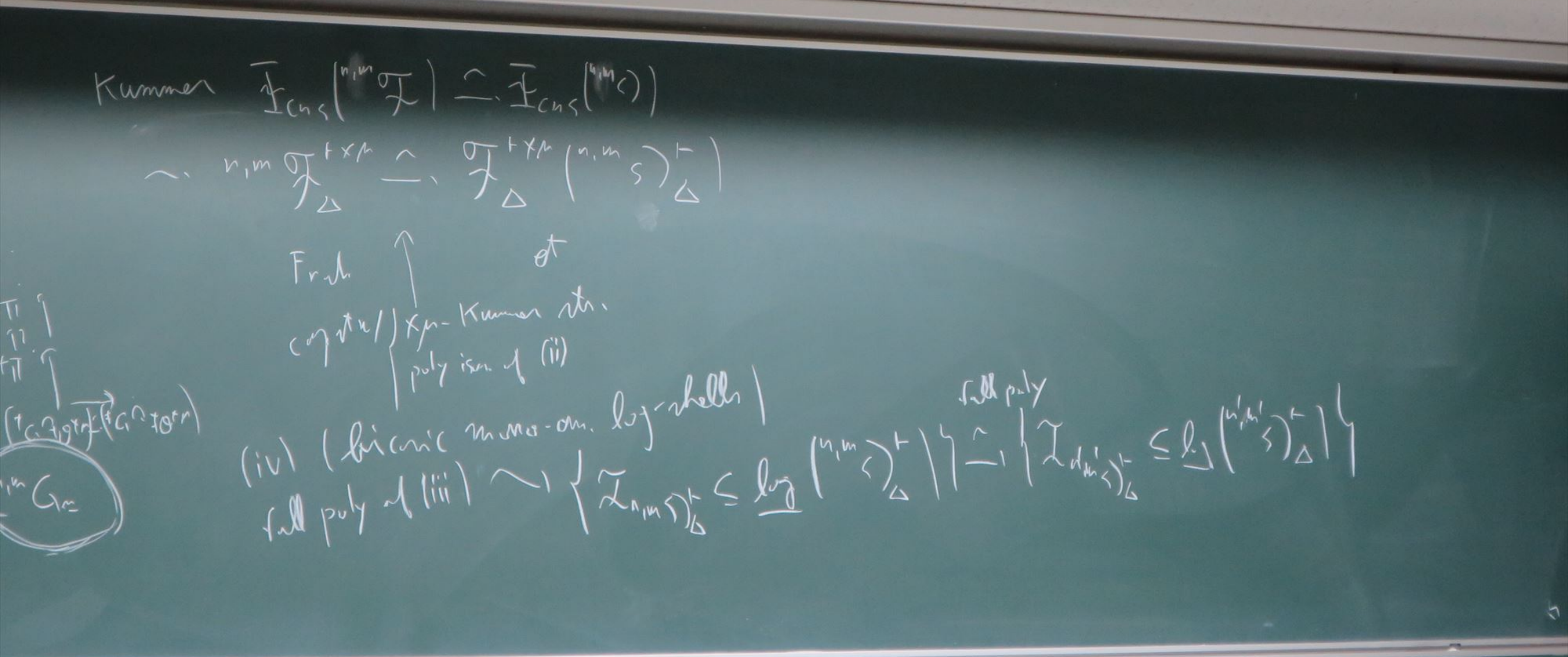
$\wr$

${}^{\ddagger}\Pi$

$$\overset{\frown}{\text{(compose)}} \quad \mathcal{F}^{\vdash\times\mu}_{\triangle}({}^{n,m}\mathfrak{S}^{\vdash}_{\triangle})| \xrightarrow[\text{full poly}]{\sim} \mathcal{F}^{\vdash\times\mu}_{\triangle}({}^{n',m'}\mathfrak{S}^{\vdash}_{\triangle})|$$

$$({}^{\dagger}\mathcal{G} \supseteq {}^{\dagger}\theta^{\times\mu}) \longrightarrow ({}^{\ddagger}\mathcal{G} \supseteq {}^{\ddagger}\theta^{\times\mu})$$

${}^{n,m}\mathbb{G}_{m}$

Kummer $\overline{\Psi}_{\mathrm{cns}}({}^{n,m}\mathfrak{F}) \overset{\sim}{\to} \overline{\Psi}_{\mathrm{cns}}({}^{n,m}\mathfrak{C}))$
$\rightsquigarrow {}^{n,m}\mathfrak{F}^{\vdash\times\mu}_{\triangle} \overset{\sim}{\to} \mathfrak{F}^{\vdash\times\mu}_{\triangle}({}^{n,m}\mathfrak{S}^{\vdash}_{\triangle}))$
Frob. $\uparrow$ ét
(cy $\dagger n$/) ×μ-Kummer str.
poly isom. of (ii)
$\dagger\Pi$
$\dagger\Pi$
$\dagger\Pi$
$(\dagger G, \dagger\mathfrak{F}_{\mathrm{gtr}}\xrightarrow{} \dagger\mathfrak{F}(\dagger G \subseteq \dagger\Theta^{\mathrm{nr}}))$
$\boxed{{}^{n,m}G_{\underline{v}}}$
(iv) (bicoric mono-an. log-shells)
full poly of (iii) $\rightsquigarrow \{\mathcal{I}({}^{n,m}\mathfrak{S}^{\vdash}_{\triangle}) \subseteq \underline{\log}({}^{n,m}\mathfrak{S}^{\vdash}_{\triangle})\} \overset{\text{full poly}}{\overset{\sim}{\to}} \{\mathcal{I}({}^{n',m'}\mathfrak{S}^{\vdash}_{\triangle}) \subseteq \underline{\log}({}^{n',m'}\mathfrak{S}^{\vdash}_{\triangle})\}$

$(\mathcal{F} - \Theta$

$$\{\mathcal{I}_{{}^{\dagger}\mathcal{F}^{\vdash\times\mu}_{\triangle}({}^{n,m}\mathfrak{S})^{\vdash}_{\triangle}} \subseteq \underline{\log}({}^{\dagger}\mathcal{F}^{\vdash\times\mu}_{\triangle}({}^{n,m}\mathfrak{S})^{\vdash}_{\triangle})\}$$

$$\xrightarrow{\sim} \{\mathcal{I}_{{}^{\dagger}\mathcal{F}^{\vdash\times\mu}_{\triangle}({}^{n',m'}\mathfrak{S})^{\vdash}_{\triangle}} \subseteq \underline{\log}({}^{\dagger}\mathcal{F}^{\vdash\times\mu}_{\triangle}({}^{n',m'}\mathfrak{S})^{\vdash}_{\triangle})\}$$

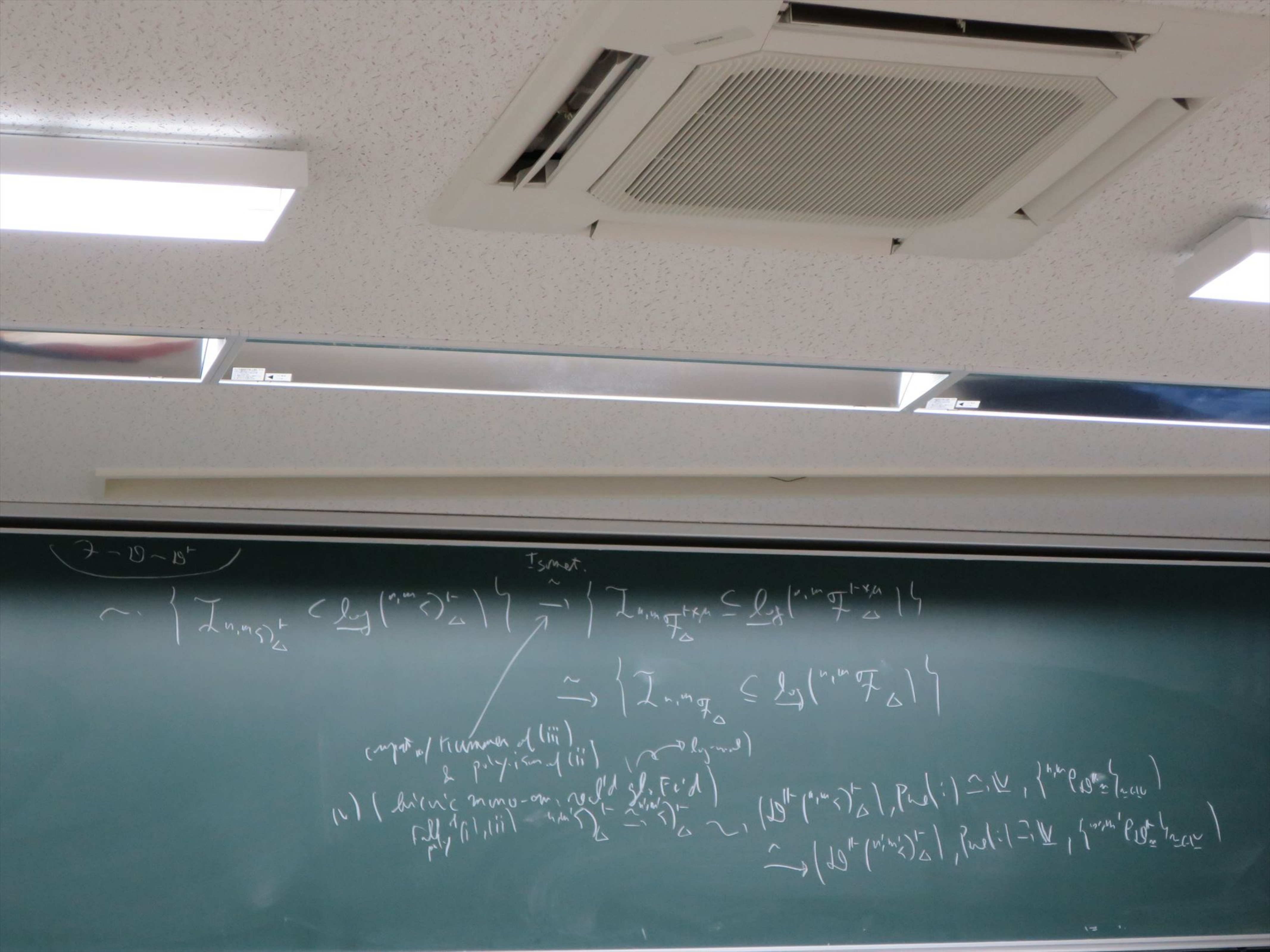
compat. w/ isomorphisms

$$\Psi_{\mathrm{cns}}({}^{\dagger}\mathcal{F}_{\succ})|_{0},\ \Psi_{\mathrm{cns}}({}^{\dagger}\mathcal{F}_{\succ})|_{\langle \mathbb{F}^{*}_{l} \rangle} \to \triangle$$

$$\rightsquigarrow \mathcal{I}_{{}^{n,m}\mathcal{F}_{\triangle}} \subseteq \underline{\log}({}^{n,m}\mathcal{F}_{\triangle})$$

Frob.-like

holo. log-shell



$\mathcal{F} \sim \mathcal{D} \sim \mathcal{D}^{\vdash}$

$\curvearrowright \{ \mathcal{I}_{{}^{n,m}\mathcal{S}^{\vdash}_{\triangle}} \subseteq \underline{\log}({}^{n,m}\mathcal{S})^{\vdash}_{\triangle} \} \overset{\text{Isomet.}}{\xrightarrow{\sim}} \{ \mathcal{I}_{{}^{n,m}\mathcal{F}^{\vdash\times\mu}_{\triangle}} \subseteq \underline{\log}({}^{n,m}\mathcal{F}^{\vdash\times\mu}_{\triangle}) \}$

$\xrightarrow{\sim} \{ \mathcal{I}_{{}^{n,m}\mathcal{F}_{\triangle}} \subseteq \underline{\log}({}^{n,m}\mathcal{F}_{\triangle}) \}$

compat. w/ Kummer of (iii) & poly-isom of (ii)

(v) (bi-coric mono-an. vert'd sh. F'd) (↷ log-mod)

fully poly (i),(iii) ${}^{n,m}\mathcal{S})^{\vdash}_{\triangle} \xrightarrow{\sim} {}^{n',m'}\mathcal{S})^{\vdash}_{\triangle} \rightsquigarrow (\mathcal{D}^{\Vdash}({}^{n,m}\mathcal{S})^{\vdash}_{\triangle}), \mathrm{Prime}(\cdot) \xrightarrow{\sim} \underline{\mathbb{V}}, \{{}^{n,m}\rho_{\mathcal{D}^{\Vdash}_{\underline{v}}}\}_{\underline{v}\in\underline{\mathbb{V}}})$

$\xrightarrow{\sim} (\mathcal{D}^{\Vdash}({}^{n',m'}\mathcal{S})^{\vdash}_{\triangle}), \mathrm{Prime}(\cdot) \xrightarrow{\sim} \underline{\mathbb{V}}, \{{}^{n',m'}\rho_{\mathcal{D}^{\Vdash}_{\underline{v}}}\}_{\underline{v}\in\underline{\mathbb{V}}})$

compat w/ $\mathbb{R}_{>0}$-orbit of isom cl

$$\left({}^{n,m}\mathfrak{C}^{\Vdash}_{\triangle},\ \mathrm{Prm}(-) \cong \underline{\mathbb{V}},\ \{ {}^{n,m}\mathfrak{C}_{\triangle,\underline{v}} \}_{\underline{v} \in \underline{\mathbb{V}}} \right)$$

$$\overset{\sim}{\longrightarrow},\ \left({}^{n',m'}\mathfrak{C}^{\Vdash}_{\triangle},\ \cdots\cdots\cdots \right)$$

w.r.t. full poly-isom of (ii)

§12.

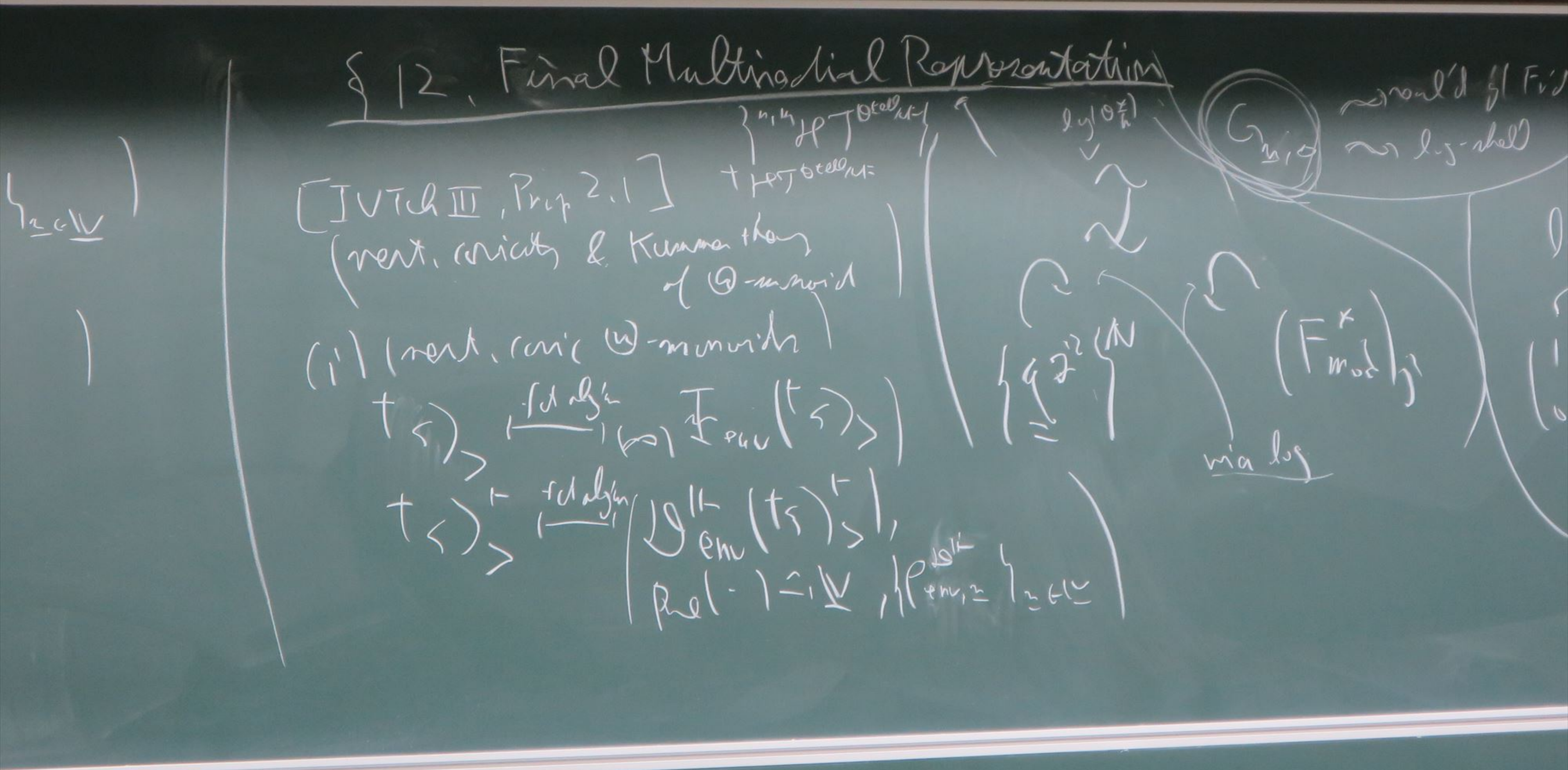
[IUTch III

(rest. con

(i') (rest

$t_{<})_{>}$

$t_{<})$



$\}_{\underline{v} \in \underline{\mathbb{V}}})$

§12. Final Multiradial Representation

$\{^{n,m}\mathcal{HT}^{\Theta^{\mathrm{ell}}\mathrm{NF}}\}$

$^{\dagger}\mathcal{HT}^{\Theta^{\mathrm{ell}}\mathrm{NF}}$

[IUTch III, Prop 2.1]
(vert. coricity & Kummer theory of Θ-monoid

(i') (vert. coric Θ-monoids

$^{\dagger}\mathcal{D}_{>} \xmapsto{\text{fct. alg'm}} (\infty)\Psi_{\mathrm{env}}(^{\dagger}\mathcal{D}_{>})$)

$^{\dagger}\mathcal{D}^{\vdash}_{>} \xmapsto{\text{fct. alg'm}} (\underline{\underline{\theta}}^{\iota}_{\mathrm{env}}(^{\dagger}\mathcal{D}^{\vdash}_{>}),$
$\mathrm{Prime}(-) \xrightarrow{\sim} \underline{\mathbb{V}}, \{\rho^{\underline{\underline{\theta}}^{\iota}}_{\mathrm{env},\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}})$

$\log(\mathcal{O}^{\times}_{\mu})$

$\wr$

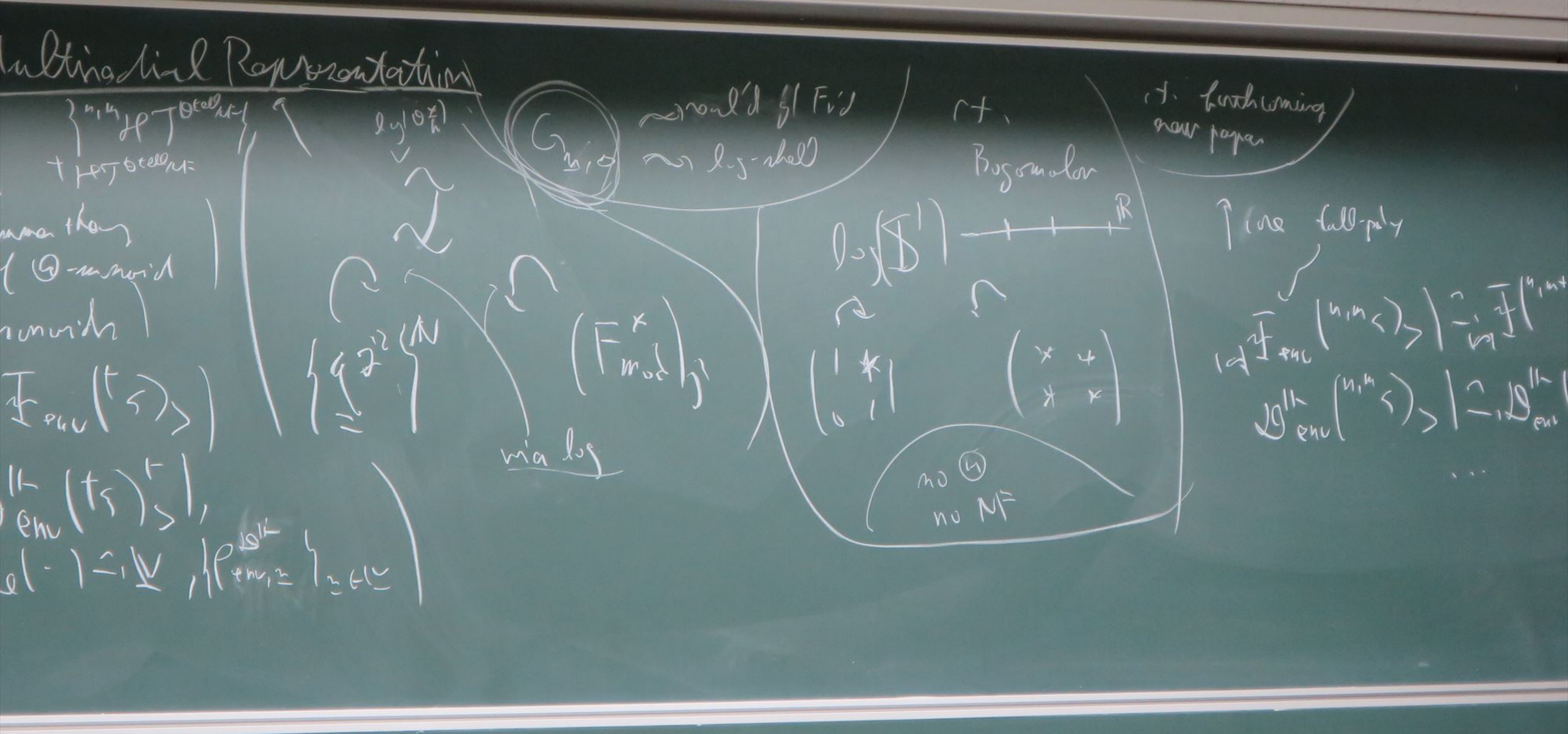
$\{q^{j^2}_{\underline{v}}\}$ $\curvearrowleft$ N

$(F^{\times}_{\mathrm{mod}})_{j}$

via log

$G_{\underline{v}}$ ⤳ recon'd of (Fr

⤳ log-shell



ultiradial Representation

$\{{}^{n,m}\mathcal{HT}^{\Theta^{\text{ell}}NF}\}$

$+ {}^{\dagger}\mathcal{HT}^{\Theta^{\text{ell}}NF}$

mmer theory

(Θ-monoid

monoids

$\mathcal{F}_{\text{env}}({}^{\dagger}\mathfrak{S})_{>}$

$\mathcal{D}^{\Vdash}_{\text{env}}({}^{\dagger}\mathfrak{S})^{\vdash}_{>}$,

$\ell(-) \cong \underline{\mathbb{V}}, \{\rho^{\mathcal{D}^{\Vdash}}_{\text{env},\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$

$\log(\Theta^{\times}_{\mu})$

$\downarrow$

$\wr$

$\{\underline{q}^{j^2}\}_{\mathbb{N}}$

$(\mathbb{F}^{\times}_{\text{mod}})_{j}$

via log

$G_{\underline{v}}$ ⟲ ~ ind'd of F_v's

~ log-shell

cf. Bogomolov

$\log(|\mathbb{S}^1|)$ — $\mathbb{R}$

$\begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix}$ $\begin{pmatrix}* & *\\ * & *\end{pmatrix}$

no Θ

no NF

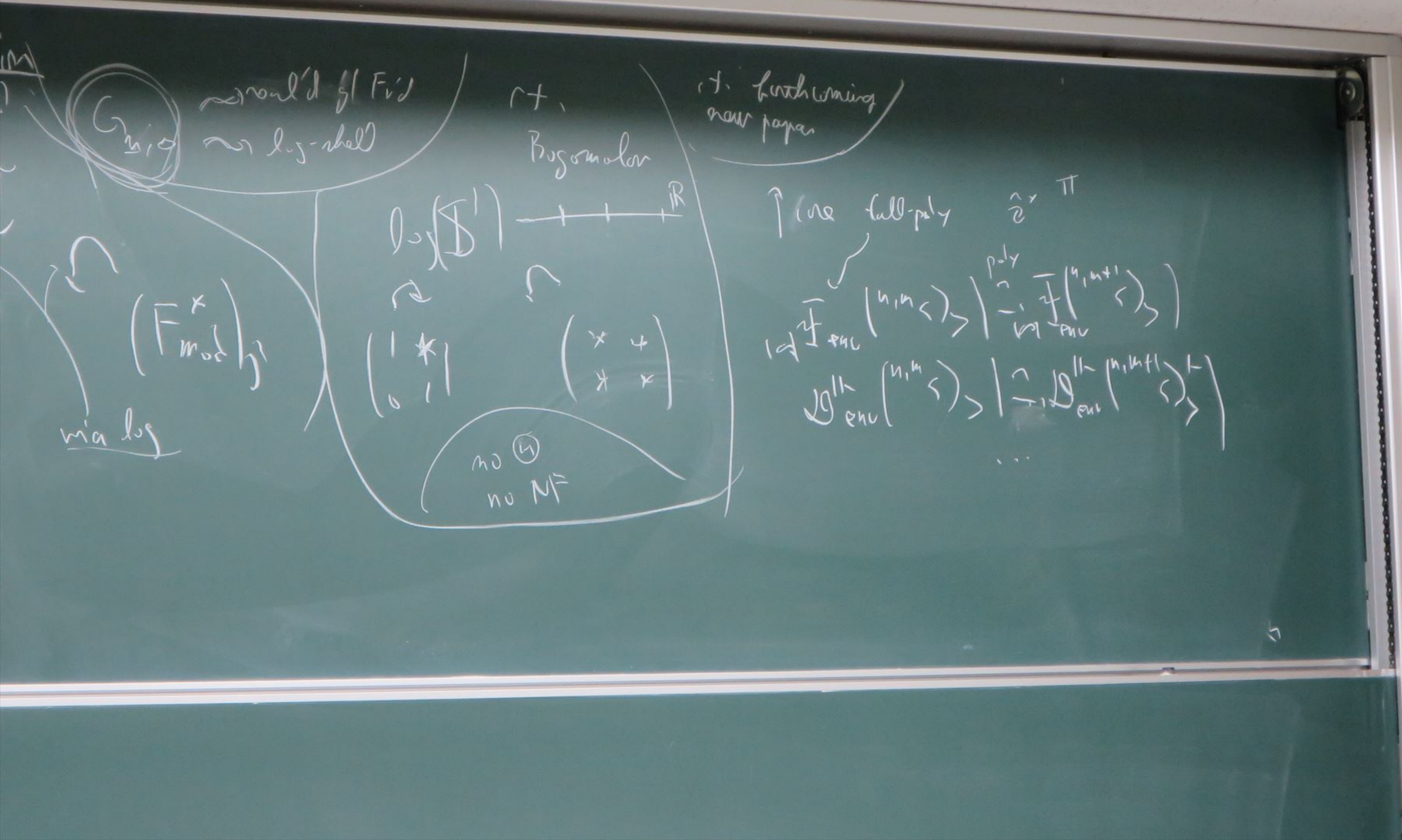
cf. forthcoming new paper

↑ (via full-poly)

$\mathcal{F}_{\text{env}}({}^{n,m}\mathfrak{S})_{>} \cong_{\text{id}} \mathcal{F}({}^{n,m+}$

$\mathcal{D}^{\Vdash}_{\text{env}}({}^{n,m}\mathfrak{S})_{>} \cong \mathcal{D}^{\Vdash}_{\text{env}}$

...



$G_{\underline{v}}, \circlearrowleft$ $\rightsquigarrow$ anab'd gl Fr'd

$\rightsquigarrow$ log-shell

$\bigcap$

$(\mathbb{F}^{\times}_{\mathrm{mod}})_{j}$

via log

cf. Bogomolov

$\log(\mathbb{S}^1)$ $\mathbb{R}$

$\circlearrowright$ $\bigcap$

$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} * & * \\ * & * \end{pmatrix}$

no ④

no NF

cf. forthcoming new paper

$\uparrow$ one full-poly $\hat{\mathbb{Z}}^{\times}$ π

$\mathrm{Id} \curvearrowright \overline{\mathcal{F}}_{\mathrm{env}}(^{n,m}\mathfrak{S}_{\succ}) \Big| \overset{\mathrm{poly}}{\underset{\mathrm{full}}{\cong}} \overline{\mathcal{F}}_{\mathrm{env}}(^{n,m+1}\mathfrak{S}_{\succ}) \Big|$

$\mathcal{D}^{\Vdash}_{\mathrm{env}}(^{n,m}\mathfrak{S}_{\succ}) \Big| \overset{\sim}{\to} \mathcal{D}^{\Vdash}_{\mathrm{env}}(^{n,m+1}\mathfrak{S}_{\succ})^{\vdash} \Big|$

$\ldots$

(ii) (Kummer isom's)

$$^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \longmapsto {}_{(\infty)}\underline{\overline{\Psi}}_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta}),$$

$$\mathcal{D}^{\vdash}_{\mathrm{env}}({}^{\dagger}\mathcal{HT}^{\Theta})$$

$$\mathrm{Prime}(\mathcal{D}^{\vdash}_{\mathrm{env}}({}^{\dagger}\mathcal{HT}^{\Theta})) \xrightarrow{\sim} \mathbb{V},$$

$$\{p^{\cdot\cdot}_{\cdot,\underline{v}}\}$$

& Kummer isom's $${}_{(\infty)}\underline{\overline{\Psi}}_{\mathcal{F}_{\mathrm{env}}}({}^{\dagger}\mathcal{HT}^{\Theta})^{\mathrm{Frob}} \xrightarrow{\sim} {}_{(\infty)}\underline{\overline{\Psi}}^{\text{ét}}_{\mathrm{env}}({}^{\dagger}\mathcal{D}_{>})$$

$$\mathcal{D}^{\vdash}_{\mathrm{env}}({}^{\dagger}\mathcal{HT}^{\Theta}) \xrightarrow{\sim} \mathcal{D}^{\vdash}_{\mathrm{env}}({}^{\dagger}\mathcal{D}^{\vdash}_{>})$$

compat w/ $\mathrm{Prime}(\ ) \xrightarrow{\sim} \underline{\mathbb{V}}$ & glob. to loc.

$\mathcal{F}^{\vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{S}_{>})$

$\mathcal{F}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{S}_{>})$

Kummer $\rightsquigarrow$

$$\left\{\begin{array}{l} {}^{\dagger}\overset{\text{Frd}}{\mathcal{F}^{\vdash}_{\mathrm{env}}} \xrightarrow{\sim} \overset{\text{ét}}{\mathcal{F}^{\vdash}_{\mathrm{env}}}({}^{\dagger}\mathfrak{S}_{>}) \\ {}^{\dagger}\mathcal{F}^{\Vdash}_{\mathrm{env}} \xrightarrow{\sim} \mathcal{F}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{S}_{>}) \end{array}\right.$$

$\xrightarrow{\sim}_{\mathrm{id}} \overset{\text{ét}}{\underline{\overline{\mathcal{F}}}}_{\mathrm{env}}({}^{\dagger}\mathfrak{S}_{>})$

$\xrightarrow{\sim} \mathcal{E}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{S})^{\kappa}_{>}$

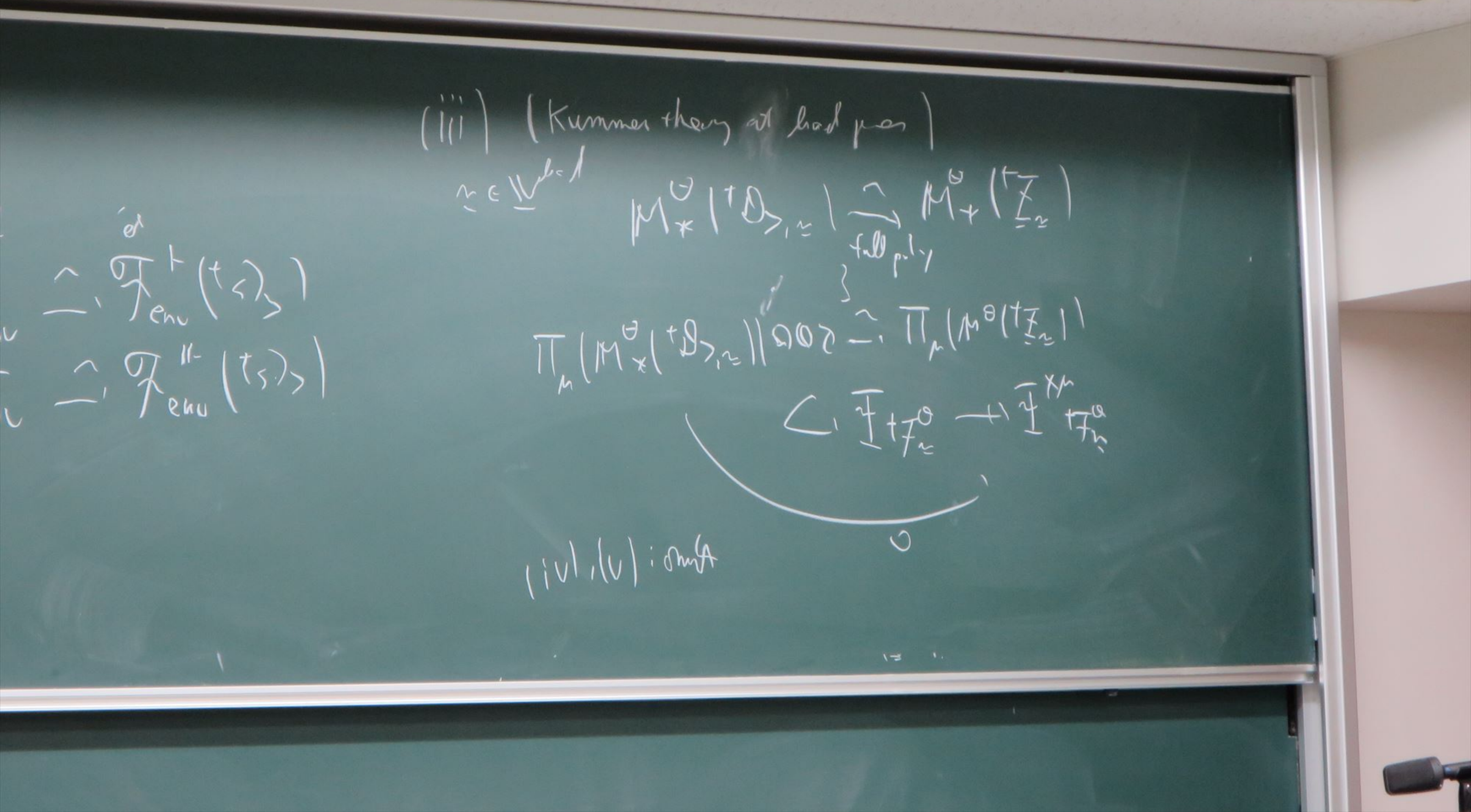
glob. loc.

(iii) (Kum

$\underline{\underline{v}} \in \underline{\underline{\mathbb{V}}}^{\mathrm{bad}}$

$\Pi_{\underline{v}}($

(iv)



$\overset{\sim}{\longrightarrow} \mathfrak{F}^{\vdash \text{el}}_{\text{env}}({}^\dagger\mathcal{D}_{>})$

$\overset{\sim}{\longrightarrow} \mathfrak{F}^{\Vdash}_{\text{env}}({}^\dagger\mathcal{D}_{>})$

(iii) (Kummer theory at bad primes)

$\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$

$$\mathcal{M}^{\Theta}_{*}({}^\dagger\mathcal{D}_{>,\underline{v}}) \xrightarrow[\text{full poly}]{\sim} \mathcal{M}^{\Theta}_{*}({}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}})$$

$$\Pi_{\mu}(\mathcal{M}^{\Theta}_{*}({}^\dagger\mathcal{D}_{>,\underline{v}})) \otimes \mathbb{Q}/\mathbb{Z} \xrightarrow{\sim} \Pi_{\mu}(\mathcal{M}^{\Theta}({}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}}))$$

$$\subseteq \Psi_{{}^\dagger\mathcal{F}^{\Theta}_{\underline{v}}} \to \Psi^{\times\mu}_{{}^\dagger\mathcal{F}^{\Theta}_{\underline{v}}}$$

$\circ$

(iv), (v) : omit

(vi) (compat. w/ const. monoids)

unit portion of the theta monoids

Frob

$$\sim\ \left\{\begin{array}{l} {}^{\dagger}\mathcal{F}^{\vdash\times}_{\Delta} \overset{\sim}{=} {}^{\dagger}\mathcal{F}^{\vdash\times}_{\mathrm{env}} \\ \mathcal{F}^{\vdash\times}_{\Delta}({}^{\dagger}\mathfrak{S})^{\dagger}_{\Delta}) \overset{\sim}{=} \mathcal{F}^{\vdash\times}_{\mathrm{env}}({}^{\dagger}\mathfrak{S}_{>}) \end{array}\right) \ \text{ét}$$

$$\left(\quad {}^{\dagger}\mathcal{F}^{\vdash\times\mu}_{\mathrm{env}\,\Delta} \overset{\sim}{=} \mathcal{F}^{\vdash\times\mu}_{\mathrm{env}\,\Delta}({}^{\dagger}\mathfrak{S}_{>}) \right) \ \text{ét}$$

$(\ )^{\times\mu}$ Kummer compatibility

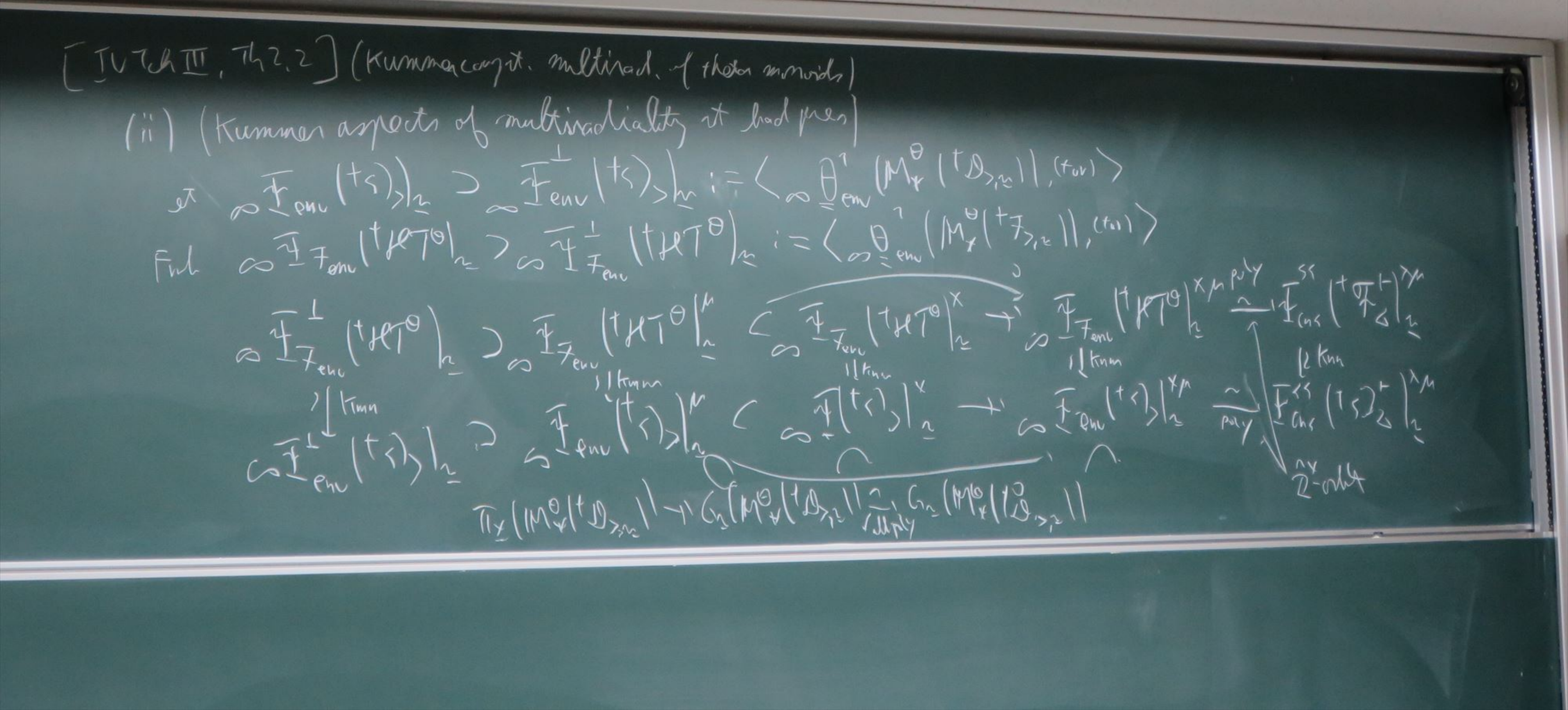
Frob

[IUTch III, Th

(ii) (Kum

ét

Frob



[IUTch III, Th 2.2] (Kummer-compat. multirad. of theta monoids)

(ii) (Kummer aspects of multiradiality at bad primes)

$$\text{et}\quad {}_\infty\underline{\overline{\Psi}}_{\mathrm{env}}({}^\dagger\mathfrak{S}_>)|_{\underline{v}} \supset {}_\infty\underline{\overline{\Psi}}^{\perp}_{\mathrm{env}}({}^\dagger\mathfrak{S}_>)|_{\underline{v}} := \langle {}_\infty\underline{\underline{\theta}}^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}({}^\dagger\mathcal{D}_{>,\underline{v}})), (\mathrm{tor}) \rangle$$

$$\text{Frd}\quad {}_\infty\underline{\overline{\Psi}}_{\mathcal{F}_{\mathrm{env}}}({}^\dagger\mathcal{HT}^{\Theta})|_{\underline{v}} \supset {}_\infty\underline{\overline{\Psi}}^{\perp}_{\mathcal{F}_{\mathrm{env}}}({}^\dagger\mathcal{HT}^{\Theta})|_{\underline{v}} := \langle {}_\infty\underline{\underline{\theta}}^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta}_{*}({}^\dagger\mathcal{F}_{>,\underline{v}})), (\mathrm{tor}) \rangle$$

$$\begin{tikzcd}
{}_\infty\underline{\overline{\Psi}}^{\perp}_{\mathcal{F}_{\mathrm{env}}}({}^\dagger\mathcal{HT}^{\Theta})|_{\underline{v}} \arrow[d, "\wr", "\mathrm{Kmn}"'] \arrow[r, phantom, "\supset"] & {}_\infty\underline{\overline{\Psi}}_{\mathcal{F}_{\mathrm{env}}}({}^\dagger\mathcal{HT}^{\Theta})|^{\mu}_{\underline{v}} \arrow[d, "\wr", "\mathrm{Kmn}"'] \arrow[r, phantom, "\subset"] \arrow[rr, bend left] & {}_\infty\underline{\overline{\Psi}}_{\mathcal{F}_{\mathrm{env}}}({}^\dagger\mathcal{HT}^{\Theta})|^{\times}_{\underline{v}} \arrow[d, "\wr", "\mathrm{Kmn}"'] \arrow[r, two heads] & {}_\infty\underline{\overline{\Psi}}_{\mathcal{F}_{\mathrm{env}}}({}^\dagger\mathcal{HT}^{\Theta})|^{\times\mu}_{\underline{v}} \arrow[d, "\wr", "\mathrm{Kmn}"'] \arrow[r, "\sim", "\mathrm{poly}"'] & {}^\dagger\underline{\overline{\Psi}}^{ss}_{\mathrm{cns}}({}^\dagger\mathcal{F}^{\vdash}_{\Delta})|^{\times\mu}_{\underline{v}} \arrow[d, "\wr", "\mathrm{Kmn}"'] \\
{}_\infty\underline{\overline{\Psi}}^{\perp}_{\mathrm{env}}({}^\dagger\mathfrak{S}_>)|_{\underline{v}} \arrow[r, phantom, "\supset"] & {}_\infty\underline{\overline{\Psi}}_{\mathrm{env}}({}^\dagger\mathfrak{S}_>)|^{\mu}_{\underline{v}} \arrow[r, phantom, "\subset"] \arrow[rr, bend right] \arrow[loop below] & {}_\infty\underline{\overline{\Psi}}({}^\dagger\mathfrak{S}_>)|^{\times}_{\underline{v}} \arrow[r, two heads] \arrow[d, phantom, "\curvearrowleft"] & {}_\infty\underline{\overline{\Psi}}_{\mathrm{env}}({}^\dagger\mathfrak{S}_>)|^{\times\mu}_{\underline{v}} \arrow[r, "\sim", "\mathrm{poly}"'] \arrow[d, phantom, "\curvearrowleft"] & \underline{\overline{\Psi}}^{ss}_{\mathrm{cns}}({}^\dagger\mathfrak{S})^{\vdash}_{\Delta}|^{\times\mu}_{\underline{v}} \arrow[d] \\
 & & & & \widehat{\mathbb{Z}}^{\times}\text{-orbit} \arrow[u, bend left, start anchor=north, end anchor=west, to path={-- (\tikztotarget)}]
\end{tikzcd}$$

$$\Pi_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^\dagger\mathcal{D}_{>,\underline{v}})) \twoheadrightarrow G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^\dagger\mathcal{D}_{>,\underline{v}})) \xrightarrow[\text{full poly}]{\sim} G_{\underline{v}}(\mathbb{M}^{\Theta}_{*}({}^\dagger\mathcal{D}_{>,\underline{v}}))$$

D-map

$\rightsquigarrow$ In particular,

id. autom's of the following obj's

$\circlearrowright$ compat w/

the autom's of $\overline{\mathcal{I}}^{ss}_{\mathrm{cns}}({}^{\dagger}\mathcal{F}^{\vdash}_{\triangle})^{\times\mu}_{\geq}$

induced by any autom. of ${}^{\dagger}\mathcal{F}^{\vdash\times\mu}_{\triangle}$

multirad.

(a)$_{\underline{v}}$ ${}_{\infty}\overline{\mathcal{I}}^{\perp}_{\mathrm{env}}({}^{\dagger}\mathfrak{S})_{>}|_{\underline{v}} \supset {}_{\infty}\overline{\mathcal{I}}_{\mathrm{env}}({}^{\dagger}\mathfrak{S})_{>}|^{\mu}_{\underline{v}}$

(b)$_{\underline{v}}$ $\Pi_{\mu}(\mathbb{M}^{\Theta}_{*}({}^{\dagger}\mathcal{D}_{>,\underline{v}}))\otimes\mathbb{Q}/\mathbb{Z}$ rel. to nat. isom.

$\Pi_{\mu}(\mathbb{M}^{\Theta}_{\infty}({}^{\dagger}\mathcal{D}_{>,\underline{v}}))\otimes\mathbb{Q}/\mathbb{Z} \xrightarrow{\sim} {}_{\infty}\overline{\mathcal{I}}_{\mathrm{env}}({}^{\dagger}\mathfrak{S})_{>}|^{\mu}_{\underline{v}}$

(c)$_{\underline{v}}$ $\mathbb{M}^{\Theta}_{\infty}({}^{\dagger}\mathcal{D}_{>,\underline{v}})$

[IUTchI

$(d)_{\simeq}$ the splitting $\rightsquigarrow \tilde{\mathcal{F}}^{\perp}_{\mathrm{fan}}((t_S)_{>})|_{\simeq} \to \rightsquigarrow \tilde{\mathcal{F}}_{\mathrm{fan}}((t_S)_{>})|^{\mu}_{\simeq}$

(and at 0-labelled cp.)

$(a)_{\simeq}$
$(b)_{\simeq}$
$(c)_{\simeq}$ $(d)_{\simeq}$

$$\tilde{\mathcal{F}}^{ss}_{\mathrm{fan}}\left(t\sigma\mathcal{F}^{\perp}_{\Delta}(-)\right)|^{\times\mu}_{\simeq}$$

mono-anab.
μ
Δ

$\mathcal{F}_{\mathrm{fan}}((t_S)_{>})|^{\mu}_{\simeq}$

(R, ρ, Φ) : m

$\Updownarrow$ $\forall$

$R \times_{\rho} R)|_{,} \to (R_1, R_2$

$(R_1, R_2, \alpha)^{\simeq}$

$\Downarrow$

(R_1, R_2, α)

$(t_\varsigma)_{>}|_{\simeq} \twoheadrightarrow_{\infty} \mathcal{F}_{\mathrm{env}}(t_\varsigma)_{>}|^{\mu}_{\simeq}$

env

belled cp.)

$(a|_{\simeq}$
$(n|_{\simeq}$
$(c|_{\simeq}$ $(d|_{\simeq}$

$\mathcal{F}^{\vdash}_{\triangle}(-)|^{\times\mu}_{\simeq}$

mono-ani. μ $\triangleright$

$(\mathcal{R}, \rho, \Phi)$: mult. rad. env. [IUTch II, Ex 1.7 (iii)]

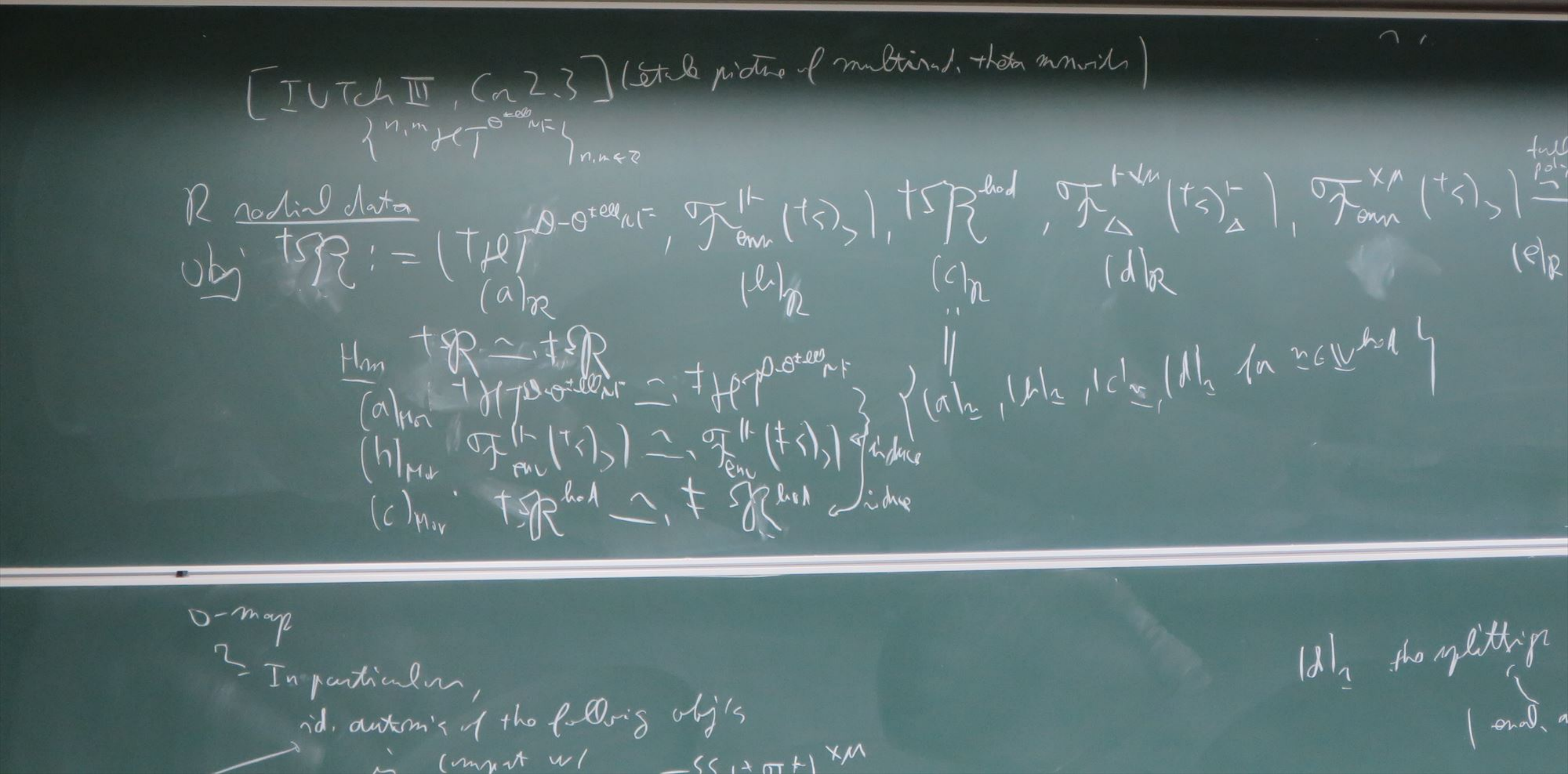
$\Updownarrow$

ob $\mathcal{R} \times_{\rho} \mathcal{R} \ni \forall (R_1, R_2, \alpha)$

$(R_1, R_2, \alpha) \simeq (R_1, R_1, \mathrm{id})$

$\Downarrow$

$(R_1, R_2, \alpha) \simeq (R_2, R_1, \alpha^{-1})$



[IUTch III, Cor 2.3] (étale picture of multiradial theta monoids)

$\{ {}^{n,m}\mathcal{HT}^{\Theta^{\pm ell}\mathrm{NF}} \}_{n,m\in\mathbb{Z}}$

$\mathcal{R}$ radial data

obj: ${}^{\dagger}\mathfrak{R} := ({}^{\dagger}\mathcal{HT}^{D\text{-}\Theta^{\pm ell}\mathrm{NF}}, {}^{\sigma}\mathcal{F}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{S})_{>}), {}^{\dagger}\mathfrak{R}^{\mathrm{mod}}, {}^{\sigma}\mathcal{F}^{\Vdash\blacktriangleright\times\mu}_{\triangle}({}^{\dagger}\mathfrak{S})^{\Vdash}_{\triangle}), {}^{\sigma}\mathcal{F}^{\times\mu}_{\mathrm{env}}({}^{\dagger}\mathfrak{S})_{>}) \overset{\text{full poly}}{\xrightarrow{\sim}}$

$(a)_{\mathcal{R}}$ $(b)_{\mathcal{R}}$ $(c)_{\mathcal{R}}$ $(d)_{\mathcal{R}}$ $(e)_{\mathcal{R}}$

$(c)_{\mathcal{R}} = \{ (a)_{\underline{v}}, (b)_{\underline{v}}, (c)_{\underline{v}}, (d)_{\underline{v}} \text{ for } \underline{v} \in \underline{\mathbb{V}}^{\mathrm{mod}} \}$

Hom ${}^{\dagger}\mathfrak{R} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{R}$

$(a)_{\mathrm{Mor}}$ ${}^{\dagger}\mathcal{HT}^{D\text{-}\Theta^{\pm ell}\mathrm{NF}} \xrightarrow{\sim} {}^{\ddagger}\mathcal{HT}^{D\text{-}\Theta^{\pm ell}\mathrm{NF}}$

$(b)_{\mathrm{Mor}}$ ${}^{\sigma}\mathcal{F}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{S})_{>}) \xrightarrow{\sim} {}^{\sigma}\mathcal{F}^{\Vdash}_{\mathrm{env}}({}^{\ddagger}\mathfrak{S})_{>})$ induce

$(c)_{\mathrm{Mor}}$ ${}^{\dagger}\mathfrak{R}^{\mathrm{mod}} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{R}^{\mathrm{mod}}$ induce

Θ-map

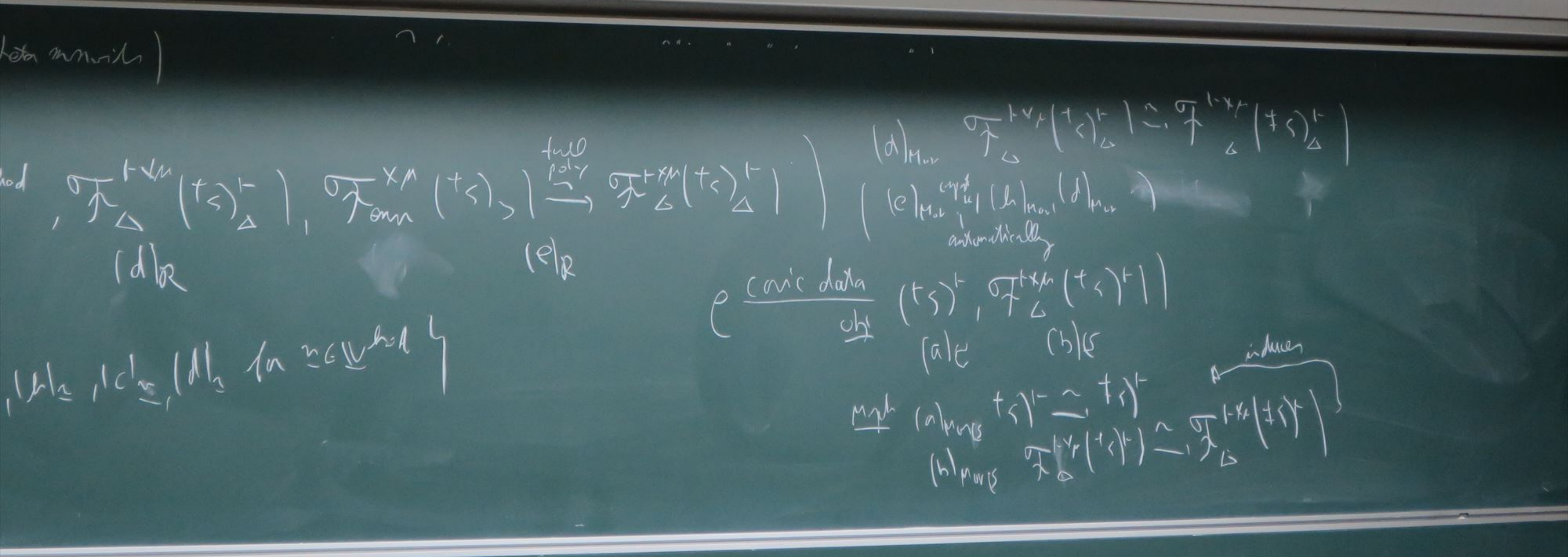
$\rightsquigarrow$ In particular,

id. autom's of the following obj's

(compat w/ $\{({}^{\dagger}\Pi_{\underline{v}})^{\times\mu}$

$(d)_{\underline{v}}$ the splittings

ond. a



data monoids)

hod , ${}^{\sigma}\mathcal{F}_{\triangle}^{\vdash\times\mu}({}^{\dagger}S)^{\vdash}_{\triangle})$, ${}^{\sigma}\mathcal{F}^{\times\mu}_{\mathrm{env}}({}^{\dagger}S)_{>}) \xrightarrow[\sim]{\text{full poly}} {}^{\sigma}\mathcal{F}^{\vdash\times\mu}_{\triangle}({}^{\dagger}S)^{\vdash}_{\triangle})$)

$(d)_{\mathbb{R}}$ $(e)_{\mathbb{R}}$

, $(b)_{\underline{n}}$, $(c)_{\underline{n}}$, $(d)_{\underline{n}}$ for $\underline{n} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ }

$(d)_{\mathrm{Mor}}$ ${}^{\sigma}\mathcal{F}^{\vdash\times\mu}_{\triangle}({}^{\dagger}S)^{\vdash}_{\triangle}) \xrightarrow{\sim} {}^{\sigma}\mathcal{F}^{\vdash\times\mu}_{\triangle}({}^{\ddagger}S)^{\vdash}_{\triangle})$

($(e)_{\mathrm{Mor}}$ $\overset{\text{compat}}{\text{w/}}$ $(b)_{\mathrm{Mor}}$, $(d)_{\mathrm{Mor}}$)
automatically

$\mathcal{E}$ $\dfrac{\text{coric data}}{\underline{\text{obj}}}$ $(({}^{\dagger}S)^{\vdash}, {}^{\sigma}\mathcal{F}^{\vdash\times\mu}_{\triangle}({}^{\dagger}S)^{\vdash}))$

$(a)_{\mathcal{E}}$ $(b)_{\mathcal{E}}$

$\underline{\text{Morph}}$ $(a)_{\mathrm{Morph}}$ $({}^{\dagger}S)^{\vdash} \xrightarrow{\sim} ({}^{\ddagger}S)^{\vdash}$ induces

$(b)_{\mathrm{Morph}}$ ${}^{\sigma}\mathcal{F}^{\vdash\times\mu}_{\triangle}(({}^{\dagger}S)^{\vdash}) \xrightarrow{\sim} {}^{\sigma}\mathcal{F}^{\vdash\times\mu}_{\triangle}(({}^{\ddagger}S)^{\vdash})$

rad. alg'm

$$^{\dagger}\mathfrak{R} = \left({}^{\dagger}\mathcal{HT}^{\Theta\text{-}\Theta^{\pm ell}NF}, {}^{\sigma}\mathcal{F}^{\Vdash}_{\mathrm{env}}({}^{\dagger}\mathfrak{S}_{>}), {}^{\dagger}\mathfrak{R}^{\mathrm{bad}}, {}^{\sigma}\mathcal{F}^{\Vdash\times\mu}_{\triangle}(({}^{\dagger}\mathfrak{S})^{\Vdash}_{\triangle}), {}^{\sigma}\mathcal{F}^{\Vdash\times\mu}_{\mathrm{env}}({}^{\dagger}\mathfrak{S}_{>}) \xrightarrow{\sim} {}^{\sigma}\mathcal{F}^{\Vdash\times\mu}_{\triangle}(({}^{\dagger}\mathfrak{S})^{\Vdash}_{\triangle})\right)$$

$$\mapsto {}^{\dagger}\mathfrak{C} = \left(({}^{\dagger}\mathfrak{S})^{\Vdash}_{\triangle}, {}^{\sigma}\mathcal{F}^{\Vdash\times\mu}_{\triangle}(({}^{\dagger}\mathfrak{S})^{\Vdash}_{\triangle})\right)$$

(i') $\uparrow$ full & ess. surj.; $\rightsquigarrow$ multirad.

(ii) $^{n,m}\mathcal{HT}^{\textcircled{\mathcal{D}}\text{-}\Theta^{\pm ell}NF}$ $\rightsquigarrow$ $^{n,m}\mathfrak{R} \xrightarrow{\sim} {}^{n,m+1}\mathfrak{R} \xrightarrow{\sim} \ldots$

$\wr\wr$

$^{n,m+1}\mathcal{HT}$

$\rightsquigarrow$ vert. coric $^{n,\circ}\mathfrak{R}$

$\rightsquigarrow$ $^{n,\circ}\mathfrak{C}$

hor. arrow: $\to {}^{n,m}\mathfrak{S})^{\Vdash}_{\triangle}$

$\wr$

identify $^{n,\circ}\mathfrak{C}$

$\ldots{}^{\vdash}_{\triangle}), \; {}^{\sigma}\mathcal{F}^{\vdash\times\mu}_{\mathrm{env}}({}^{\dagger}\mathfrak{S})_{>} | \cong {}^{\sigma}\mathcal{F}^{\vdash\times\mu}_{\triangle}({}^{\dagger}\mathfrak{S})^{\vdash}_{\triangle} |)$

hor. arrow ..

$\to {}^{n,m}\mathfrak{S}^{\vdash}_{\triangle} \xrightarrow[\sim]{\text{full poly}} {}^{n+1,m}\mathfrak{S}^{\vdash}_{\triangle} \xrightarrow[\sim]{\text{full poly}} \ldots$

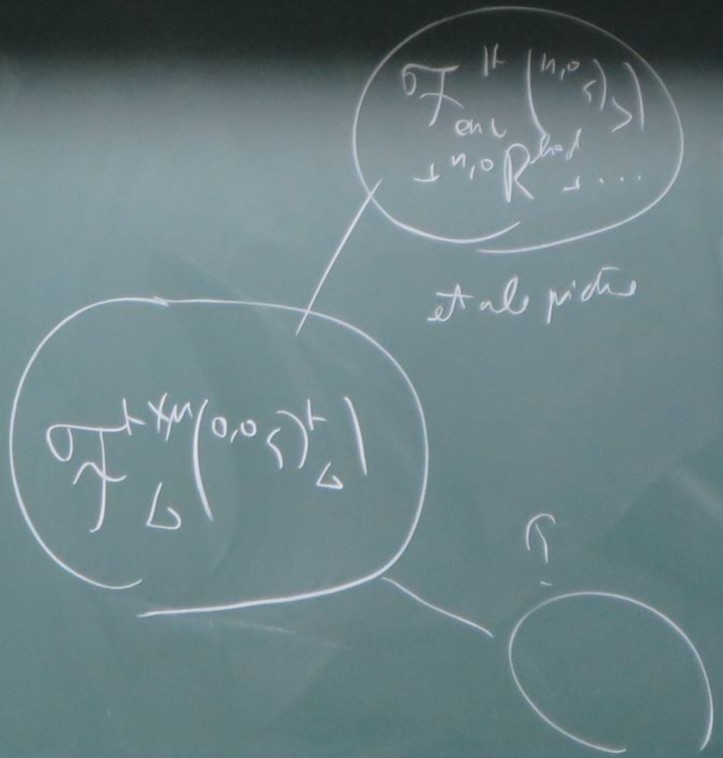
$\wr$

identify ${}^{n,\circ}\mathfrak{S}$ $|n \in \mathbb{Z}|$ $\leadsto$ ${}^{\circ,\circ}\mathfrak{S}$

$\mathcal{R} \cong \ldots$

${}^{\circ}\mathfrak{R}$

$\leadsto {}^{n,\circ}\mathfrak{S}$



étale picture

[IUTchII

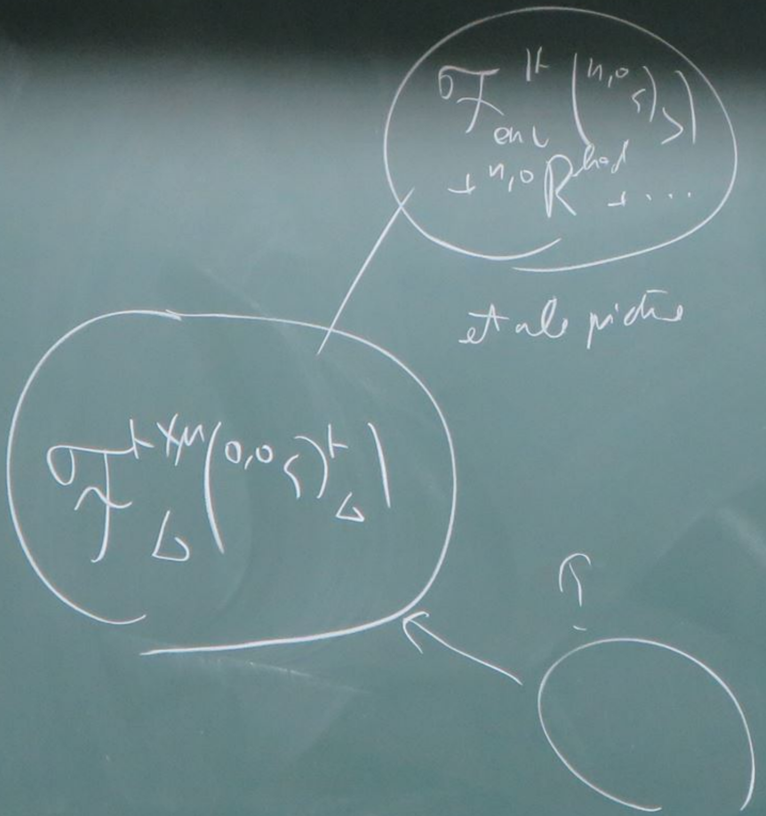
(i) tot

$\cong$

$\sim$, to

to

full
poly



[IUTch III, Def 2.4]

(i) $^{\dagger}\mathcal{F}^{\vdash} = \{^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}\}$

$\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}} \rightsquigarrow$ split Frd $^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}$

$\left(\begin{array}{l}\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \\ \mathcal{O}^{\triangleright}(-) := \mathcal{O}^{\perp}(-)\end{array}\right)$ $\quad \mathcal{O}^{\perp}(-) \subseteq \mathcal{O}^{\triangleright}(-)$, $\downarrow /(\mathrm{tors})$, $\mathcal{O}^{\circledast}(-)$

$\rightsquigarrow$ $^{\dagger}\mathcal{F}^{\vdash\perp} := \{^{\dagger}\mathcal{F}^{\vdash\perp}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ $\quad \mathcal{O}^{\triangleright} \rightsquigarrow \mathcal{O}^{\perp}$

$^{\dagger}\mathcal{F}^{\vdash\circledast} := \{^{\dagger}\mathcal{F}^{\vdash\circledast}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ $\quad \mathcal{O}^{\triangleright} \rightsquigarrow \mathcal{O}^{\circledast}$

[IUTch III, Prop 3.1]

$\mathbb{V}_{\mathbb{Q}} := \mathbb{V}(\mathbb{Q})$

@CJ

$\{{}^{\alpha}\mathcal{F}\}_{\alpha\in A}$: n-capsule of $\mathcal{F}$-prime-strip

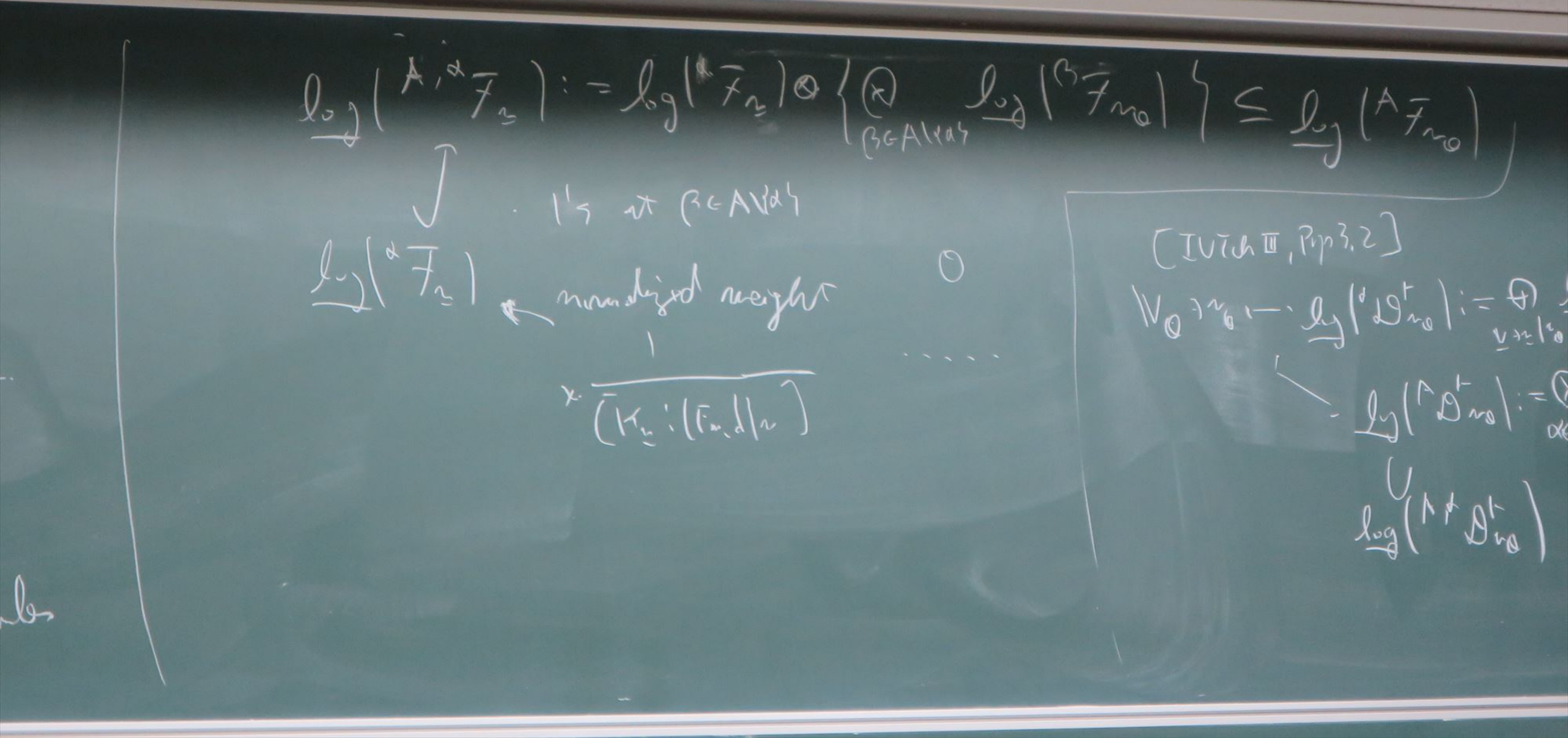
$$\mathbb{V}_{\mathbb{Q}} \ni v_{\mathbb{Q}} \mapsto \underline{\log}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}}) := \bigoplus_{\substack{\underline{v}|v_{\mathbb{Q}} \\ \underline{\mathbb{V}}}} \underline{\log}({}^{\alpha}\mathcal{F}_{\underline{v}})$$

$\underline{\text{(1-)tensor packet}}$

$$\mapsto \underline{\log}({}^{A}\mathcal{F}_{v_{\mathbb{Q}}}) := \bigotimes_{\alpha\in A} \underline{\log}({}^{\alpha}\mathcal{F}_{v_{\mathbb{Q}}})$$

$\underline{\text{(n-)tensor packet}}$

tensor as inductive limit of top. modules



$$\underline{\log}({}^{A,\alpha}\bar{\mathcal{F}}_{\underline{v}}) := \underline{\log}({}^{\alpha}\bar{\mathcal{F}}_{\underline{v}}) \otimes \left\{ \bigotimes_{\beta \in A\setminus\{\alpha\}} \underline{\log}({}^{\beta}\bar{\mathcal{F}}_{\mathrm{mod}}) \right\} \subseteq \underline{\log}({}^{A}\bar{\mathcal{F}}_{\mathrm{mod}})$$

$\cup$ · 1's at $\beta \in A\setminus\{\alpha\}$

$\underline{\log}({}^{\alpha}\bar{\mathcal{F}}_{\underline{v}})$ ← normalized weight $\times \frac{1}{[K_{\underline{v}} : (F_{\mathrm{mod}})_v]}$

$\ominus$

. . . . . .

ules

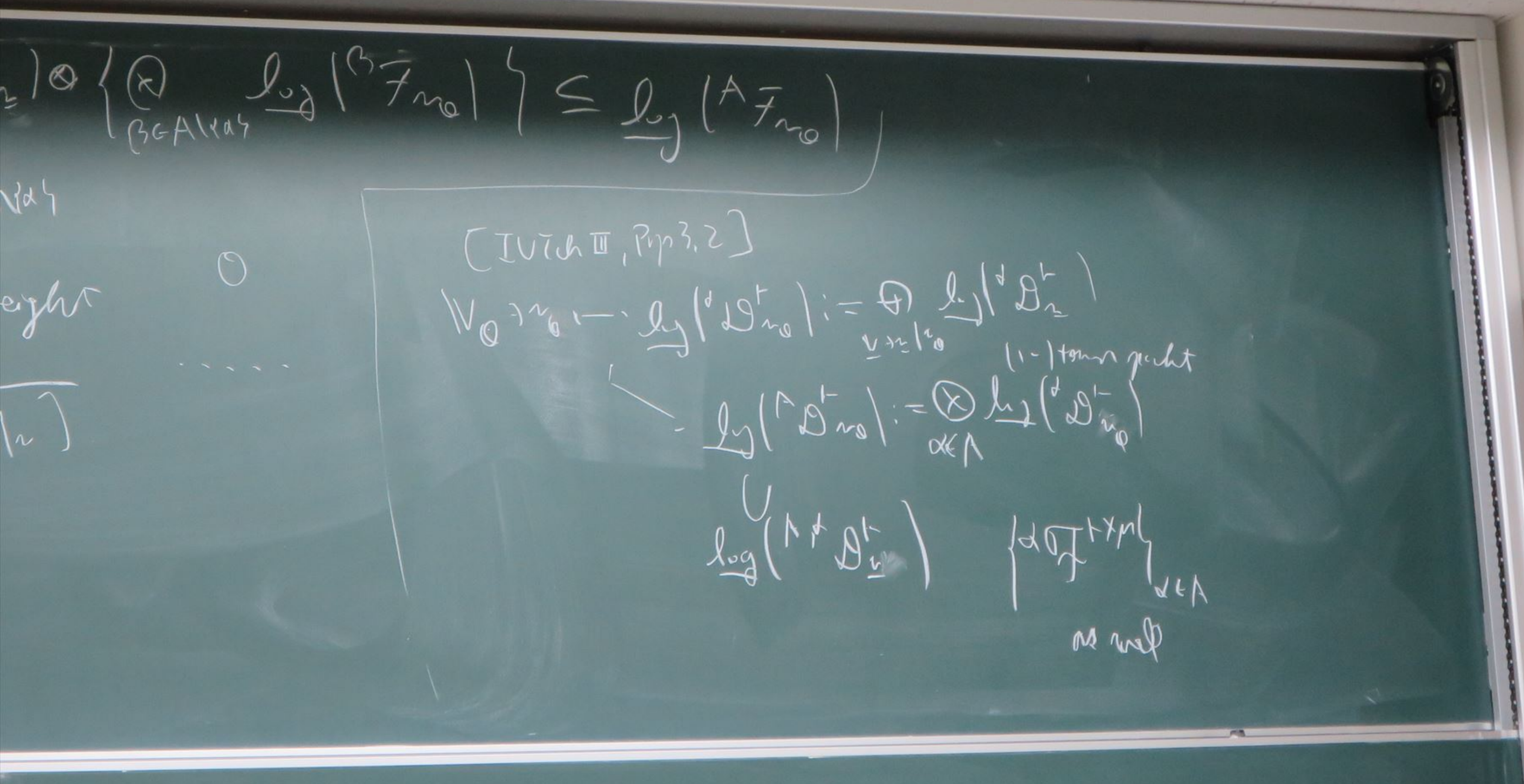
[IUTch III, Prop 3.2]

$\mathbb{V}_{\mathrm{mod}} \ni v_0 \mapsto \underline{\log}({}^{\alpha}\mathfrak{D}^{\vdash}_{v_0}) := \bigoplus_{\underline{v} \mid v_0}$

$\underline{\log}({}^{A}\mathfrak{D}^{\vdash}_{v_0}) := \bigotimes_{\alpha}$

$\cup$

$\underline{\log}({}^{A,\alpha}\mathfrak{D}^{\vdash}_{v_0})$



$\ldots) \otimes \left\{ \bigotimes_{\beta \in A \setminus \{\alpha\}} \underline{\log}({}^{\beta}\mathcal{F}_{v_\Theta}) \right\} \subseteq \underline{\log}({}^{A}\mathcal{F}_{v_\Theta})$

$\setminus\{\alpha\}$

eight

$\bigcirc$

......

[IUTch III, Prp 3.2]

$\mathbb{V}_{\Theta} \ni v_\Theta \mapsto \underline{\log}({}^{\alpha}\mathcal{D}^{\vdash}_{v_\Theta}) := \bigoplus_{\underline{v} \to v_\Theta} \underline{\log}({}^{\alpha}\mathcal{D}^{\vdash}_{\underline{v}})$

(1-)tensor packet

$\underline{\log}({}^{A}\mathcal{D}^{\vdash}_{v_\Theta}) := \bigotimes_{\alpha\in A} \underline{\log}({}^{\alpha}\mathcal{D}^{\vdash}_{v_\Theta})$

$\cup$

$\underline{\log}({}^{A,\alpha}\mathcal{D}^{\vdash}_{\underline{v}})$

$\{{}^{\alpha}\mathcal{F}^{\vdash\times\mu}\}_{\alpha\in A}$

as well

$$\mathcal{I}\left({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}\right) \subseteq \underline{\log}\left({}^{\alpha}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}\right)$$

$$\mathcal{I}\left({}^{A}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}\right) \subseteq \underline{\log}\left({}^{A}\mathcal{D}^{\vdash}_{v_{\mathbb{Q}}}\right)$$

$$\mathcal{I}\left({}^{A,\alpha}\mathcal{D}^{\vdash}_{\underline{v}}\right) \subseteq \underline{\log}\left({}^{A,\alpha}\mathcal{D}^{\vdash}_{\underline{v}}\right)$$

mono-an. log-shells

$$\mathcal{I}$$

$$\{\underline{\underline{q}}^{j^2}\}\ \left(\Gamma \cap \left(\mathbb{F}^{\times}_{\mathrm{mod}}\right)_{j}\right.$$

$${}^{\dagger}\mathcal{HT}^{\Theta^{\pm ell}\mathrm{NF}} \sim {}^{\dagger}\mathcal{H}$$

$$\sim {}^{\dagger}\mathcal{H}$$

$$\{{}^{\dagger}$$

$$A < J$$

$$\sim$$

[IUTchII, Prop 3.1]

$\mathcal{D}^{\vdash}_{\underline{v}}$)

$\mathcal{D}^{\vdash}_{\Sigma}$)

$$\overset{\mathcal{I}}{\{\underline{\underline{q}}^{j^2}\}\left(\Gamma - \cap\ (F^{\times}_{\mathrm{mod}})_{j}\right)}$$

$${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \rightsquigarrow {}^{\dagger}\mathcal{HT}^{\Theta\mathrm{NF}}$$

$$\rightsquigarrow {}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}}$$

$$\{{}^{\dagger}\mathfrak{F}_{j}\}_{j\in J}$$

$A \subset J \rightsquigarrow \{{}^{\alpha}\mathfrak{F}\}_{\alpha\in A}$

$n := |A|$

n-capsule of $\mathcal{F}$-prime-strips

$\log$-Kum $\ {}^{\alpha}\mathfrak{F} \xrightarrow[\ell_j]{}$, $\alpha\in A$

$${}^{\dagger}\mathcal{D}^{\circledcirc}_{\mathrm{Belyi}} \rightsquigarrow {}^{\dagger}\overline{\mathbb{M}}^{\circledcirc}_{\mathrm{mod}}$$

$$({}^{\dagger}\overline{\mathbb{M}}^{\circledcirc}_{\mathrm{mod}}$$

$$(({}^{\dagger}\overline{\mathbb{M}}^{\circledcirc}_{\mathrm{mod}}))_{A}$$

$$\mathcal{B}(\underline{C}_K)^{0}$$

$$\begin{tikzcd}
{}^{\dagger}\mathcal{HT}^{\Theta NF} \arrow[d, bend left] \\
{}^{\dagger}\mathcal{HT}^{\Theta^{\pm ell}} \\
\{{}^{\dagger}\mathfrak{F}_j\}_{j\in J}
\end{tikzcd}$$

$\rightsquigarrow \{{}^{\alpha}\mathfrak{F}\}_{\alpha\in A}$

$n := |A|$

n-capsule of $\mathcal{F}$-pr-strips

$\alpha \in A$

log-link ${}^{\alpha}\mathfrak{F} \xrightarrow[\log]{} {}^{\dagger}\mathfrak{F}_{\alpha}$

${}^{\dagger}\mathcal{D}^{\circledcirc} \underset{\text{Belyi}}{\rightsquigarrow} {}^{\dagger}\overline{\mathbb{M}}^{\circledcirc}_{\mathrm{mod}}$

"$F_{\mathrm{mod}}^{\times}$"

$({}^{\dagger}\overline{\mathbb{M}}^{\circledcirc}_{\mathrm{mod}})_j$

$({}^{\dagger}\overline{\mathbb{M}}^{\circledcirc}_{\mathrm{mod}})_A := \bigotimes_{A \ni \alpha} ({}^{\dagger}\overline{\mathbb{M}}^{\circledcirc}_{\mathrm{mod}})_{\alpha}$

tensor as modules

<u>global log-packet</u>

$\mathcal{B}(\underline{C}_K)^{\circ}$

$({}^{\dagger}\overline{\mathbb{M}}_{\mathrm{mod}})_A \hookrightarrow \underline{\log}({}^{A}\mathcal{F}_{V_{\Theta}}$

$\uparrow$

after taking

$(F_{\mathrm{mod}} \leftarrow$ inv. part ,

$\overline{\quad}_{l, j}$, ${}^{\dagger\sigma}\mathcal{F}_{\alpha}$

$\overline{\mathbb{M}}^{\circledcirc}_{\mathrm{mod}}$

"$F_{\mathrm{mod}}^{\times}$"

$\overline{\mathbb{M}}^{\circledcirc}_{\mathrm{mod}})_j$

$\overline{\mathbb{M}}^{\circledcirc}_{\mathrm{mod}})_A := \bigotimes_{A \ni \alpha} ({}^{\dagger}\overline{\mathbb{M}}^{\circledcirc}_{\mathrm{mod}})_{\alpha}$

tensor as modules

<u>global Ind-packet</u>

$$({}^{\dagger}\overline{\mathbb{M}}_{\mathrm{mod}})_A \hookrightarrow \underline{\log}({}^{A}\overline{\mathcal{F}}_{\mathbb{V}_{\Theta}}) := \prod_{v_{\Theta} \in \mathbb{V}_{\Theta}} \log({}^{A}\overline{\mathcal{F}}_{v_{\Theta}})$$

$$\uparrow$$

after taking no

($F_{\mathrm{mod}} \leftarrow$ inv. part, indet.)

[IUTch III, Prp 3.4] (local packet-th'c $\hat{F}$r'ds)

(i') $\alpha \in A$, $\underline{v} \in \underline{\mathbb{V}}$, $v_0 \in \mathbb{V}_0$, $\underline{v} | v_0$

$\underline{v}$

α log

$\otimes$ log $\rightarrow$ v_0

A-holo

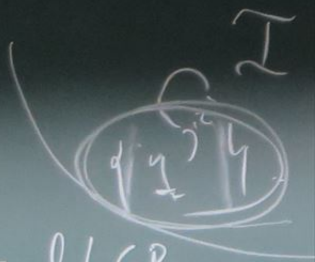
in

$$\underline{\log}({}^{\alpha}\mathcal{F}_{\underline{v}}) \hookrightarrow \underline{\log}({}^{A,\alpha}\mathcal{F}_{\underline{v}})$$

$$\text{"}\mathcal{O}^{\triangleright}_{\hat{F}_v}\text{"} \quad \mathcal{I}\log({}^{\alpha}\mathcal{F}_{\underline{v}}) \overset{\text{imag}}{\rightsquigarrow} \mathcal{I}\log({}^{A,\alpha}\mathcal{F}_{\underline{v}})$$

$$\text{"}\mathcal{O}^{\times}_{\hat{F}_v}\text{"} \quad \check{\mathcal{I}}^{\times}\log({}^{\alpha}\mathcal{F}_{\underline{v}}) \overset{\text{imag}}{\rightsquigarrow} \mathcal{I}^{\times}$$

$$\text{"}\mathbb{R}_{\geq 0}\text{"} \quad \checkmark \quad \mathcal{I}^{\mathbb{R}}\log({}^{\alpha}\mathcal{F}_{\underline{v}}) \overset{\text{only full imag}}{\rightsquigarrow} \mathcal{I}^{\mathbb{R}}$$

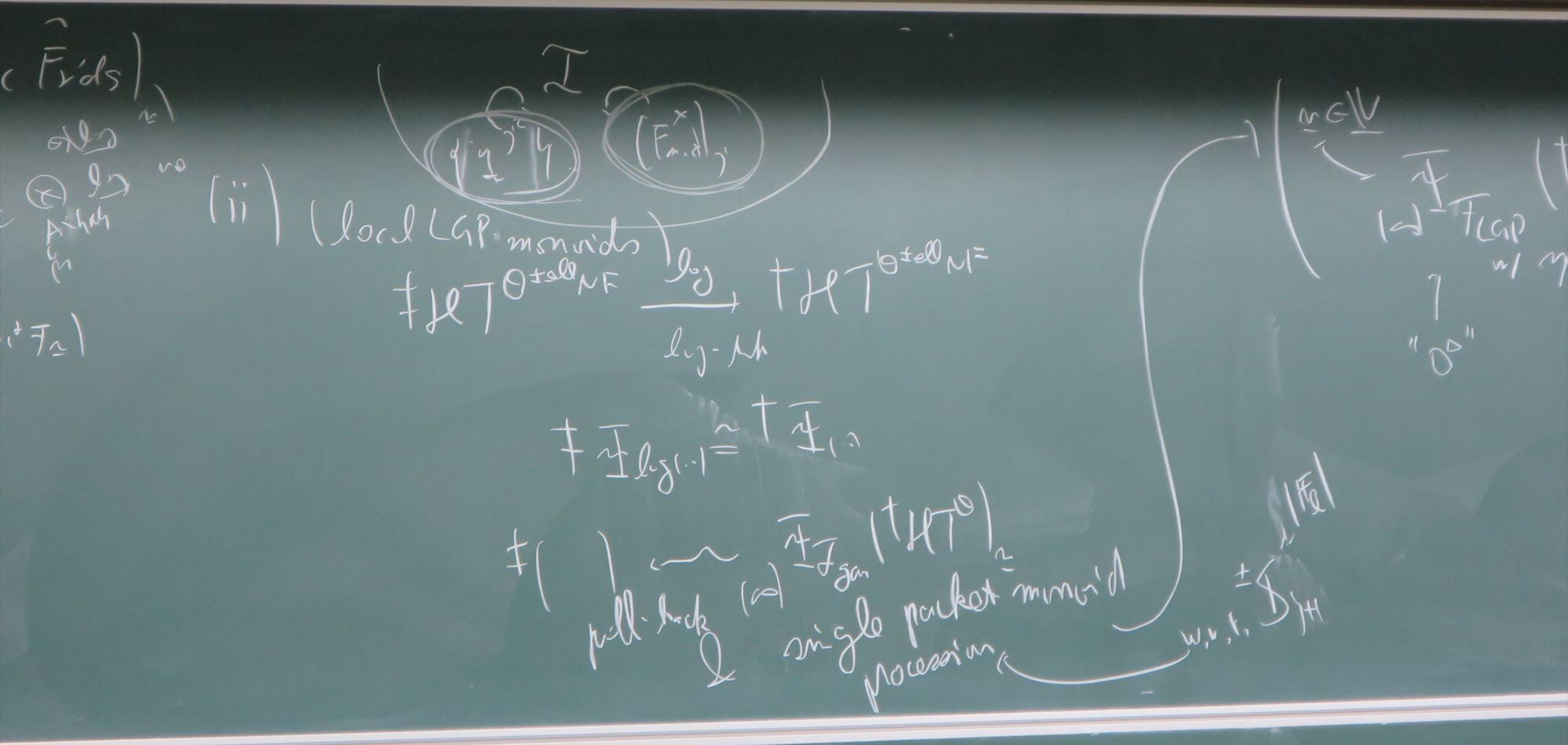


(ii) (local LGP-monoi

$\dagger\mathcal{H}\mathcal{T}^{\Theta^{\pm ell}N}$

$\dagger$

$\dagger($



$(\hat{\overline{F}}_{\mathrm{mods}})_{\simeq})$

$\xrightarrow{\log}$

⊗ $\xrightarrow{\log}$ ∞

Archimedean

$({}^{\dagger}\mathcal{F}_{\simeq})$

$\mathcal{I}$

$\overbrace{\{q_{\underline{\underline{v}}}^{j^2}\}_j}$ $\overbrace{(F^{\times}_{\mathrm{mod}})_j}$

(ii) (local LGP-monoids)

$${}^{\ddagger}\mathcal{HT}^{\Theta^{\pm ell}NF} \xrightarrow[\text{log-Kum}]{\log} {}^{\dagger}\mathcal{HT}^{\Theta^{\pm ell}NF}$$

$${}^{\ddagger}\mathcal{I}_{\log(\ldots)} \cong {}^{\dagger}\mathcal{I}_{(\ldots)}$$

$${}^{\ddagger}(\quad) \leftsquigarrow (\rightsquigarrow) \ \mathcal{I}\mathcal{F}_{\mathrm{gau}}({}^{\dagger}\mathcal{HT}^{\Theta})_{\simeq}$$

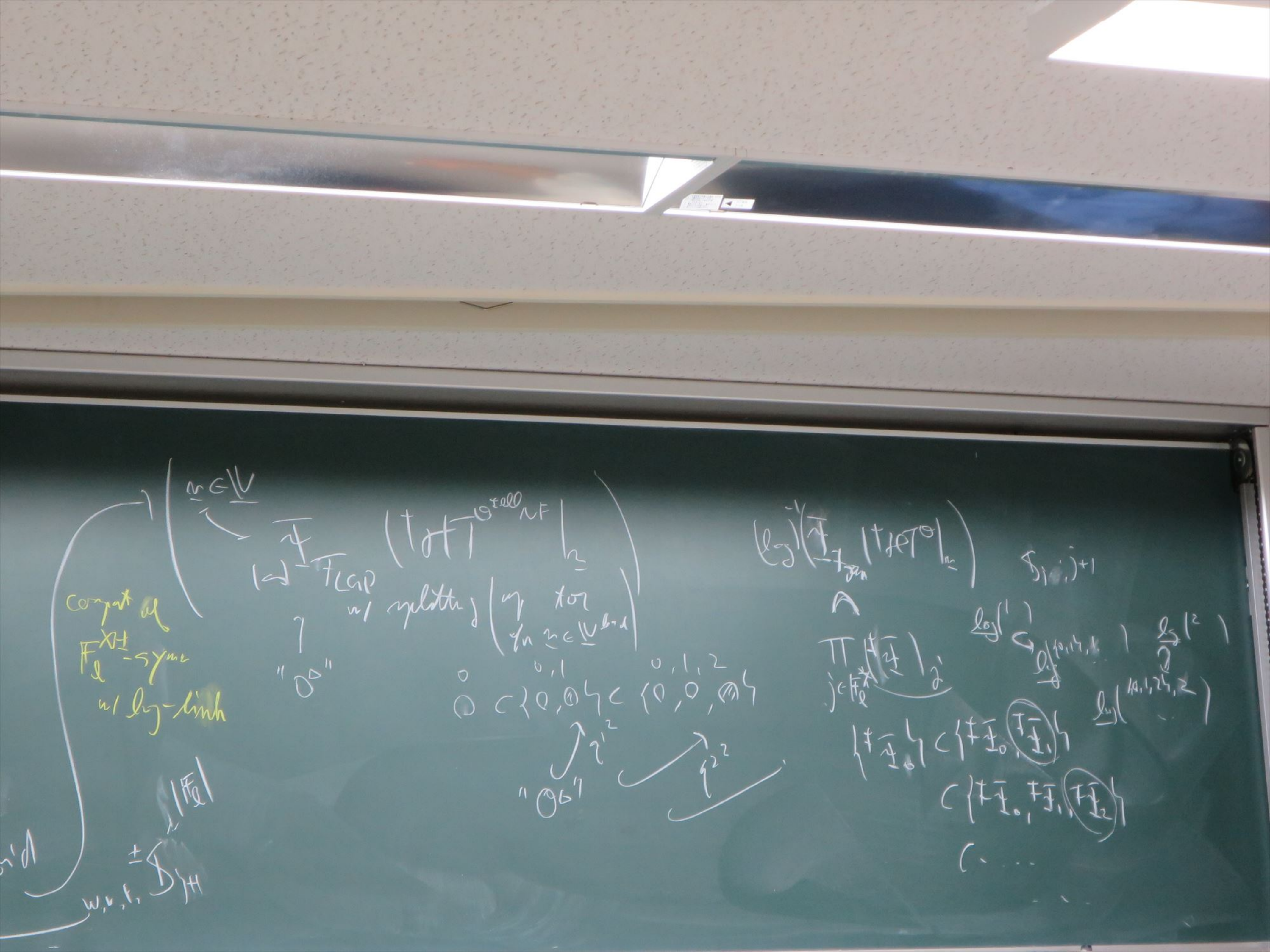
pull-back & single packet monoid processions

w.r.t. $|\mathbb{F}_l|$ ${}^{\pm}\mathfrak{S}_{j+1}$

$\underline{v} \in \underline{\mathbb{V}}$

$\mapsto \mathcal{I}$ ($\rightsquigarrow$) $\mathcal{F}_{\mathrm{LGP}}$ w/ ...

"$0^{\triangle}$"



$\underline{v} \in \underline{\mathbb{V}}$

$(\mathrm{ind})\ \mathcal{I}_{\mathcal{F}_{LGP}}\left({}^{\dagger}\mathfrak{D}\text{-}\Theta^{\pm ell}NF\right|_{\underline{v}})$

w/ splitting (up to indet. for $\underline{v} \in \underline{\mathbb{V}}^{bad}$)

$\wr$

"$\mathcal{O}^{\triangleright}$"

compat w/ $\mathbb{F}_l^{\rtimes\pm}$-symm w/ log-link

w.r.t. $^{\pm}\mathfrak{S}_{j+1}$ $|\mathbb{F}_l|$

...n'd

$\overset{0}{\mathcal{O}} \subset \{\overset{0,1}{\mathcal{O},\mathcal{O}}\} \subset \{\overset{0,1,2}{\mathcal{O},\mathcal{O},\mathcal{O}}\}$

"$\mathcal{O}^{\mu}$" $\wr^2$ $\wr^2$

$\log^{-1}\left(\mathcal{I}_{\mathcal{F}_{gau}}({}^{\dagger}\mathcal{HT}^{\Theta}|_{\underline{v}})\right)$

$\frak{S}_{j,j+1}$

$\cap$

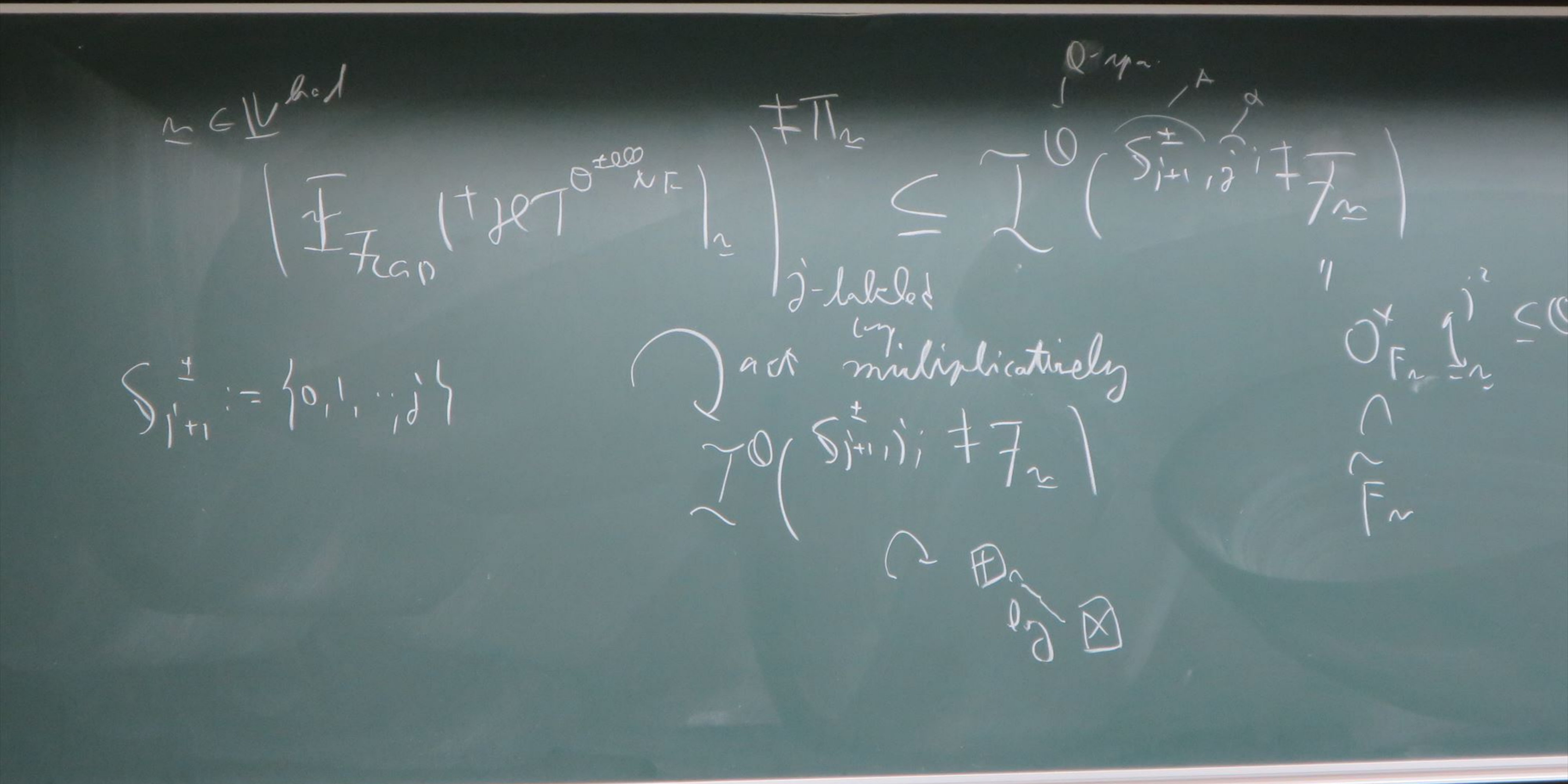
$\prod_{j\in\mathbb{F}_l^{*}} \mathcal{I}({}^{\dagger}\mathcal{I})|_j$

$\log(^{1}\ )$ $\hookrightarrow$ $\log(^{\{0,1\},1}\ )$ $\quad \log(^{2}\ )$ $\log(^{\{0,1,2\},2}\ )$

$\{{}^{\dagger}\mathcal{I}_0\} \subset \{{}^{\dagger}\mathcal{I}_0, \boxed{{}^{\dagger}\mathcal{I}_1}\}$

$\subset \{{}^{\dagger}\mathcal{I}_0, {}^{\dagger}\mathcal{I}_1, \boxed{{}^{\dagger}\mathcal{I}_2}\}$

$\subset \ldots$



$\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$

$$\left(\widetilde{\Psi}_{\mathcal{F}_{\mathrm{LGP}}}({}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}})_{\underline{v}} \right)^{\ddagger\Pi_{\underline{v}}}_{j\text{-labelled}} \subseteq \widetilde{\mathcal{I}}^{\overset{\mathbb{Q}\text{-span}}{\mathbb{Q}}}(\overset{A}{\mathbb{S}^{\pm}_{j+1}}, \overset{\alpha}{j}; {}^{\ddagger}\mathcal{F}_{\underline{v}})$$

$\mathbb{S}^{\pm}_{j+1} := \{0, 1, \ldots, j\}$

$\curvearrowright$ act (ug. multiplicatively

$\widetilde{\mathcal{I}}^{\mathbb{Q}}(\mathbb{S}^{\pm}_{j+1}, j; {}^{\ddagger}\mathcal{F}_{\underline{v}})$

$\curvearrowright \boxplus \underset{\log}{\frown} \boxtimes$

$\parallel$

$\mathcal{O}^{\times}_{F_{\underline{v}}} \cdot \underline{\underline{q}}^{j^2}_{\underline{v}} \subseteq$

$\cap$

$\widetilde{F}_{\underline{v}}$

$$\ell\ell\ell_{NF}\big|_{\underline{v}}\Big)\Big|^{\ddagger\Pi_{\underline{v}}}_{j\text{-labeled}} \subseteq \overset{\mathbb{Q}\text{-span}}{\mathcal{I}^{\mathbb{Q}}}\Big(\overset{\wedge}{\overset{\frown}{{}^{S^{\pm}_{j+1}}}},\overset{\alpha}{{}^{j}};{}^{\ddagger}\mathcal{F}_{\underline{v}}\Big)$$

$$\text{''}\quad \mathcal{O}^{\times}_{F_{\underline{v}}}\cdot q^{j^2}_{\underline{v}} \subseteq \mathbb{Q}\log(\mathcal{O}^{\times}_{F_{\underline{v}}})\quad\text{''}$$

$$\begin{array}{c}\mathcal{O}^{\times}_{F_{\underline{v}}}\\ \cap\\ \widetilde{F}_{\underline{v}}\end{array}$$

$\circlearrowright$ act $\overset{(\log.)}{\text{multiplicatively}}$

$$\mathcal{I}^{\mathbb{Q}}\Big({}^{S^{\pm}_{j+1},j;\ddagger}\mathcal{F}_{\underline{v}}\Big)$$

$$\curvearrowright \boxplus \overset{\log}{\frown} \boxtimes$$

[IUTch III

$\{$

(i

[IUTchIII, Prop 3.5] (Kummer theory & upper semi-compat. for vert. coric local LGP-monoids)

$$\{ {}^{n,m}\mathcal{HT}^{\Theta^{\pm ell}NF} \}_{n,m\in\mathbb{Z}} \underset{\text{full log-link}}{\rightsquigarrow} \underset{\text{vert. coric}}{{}^{n,\circ}\mathcal{HT}^{D\text{-}\Theta^{\pm ell}NF}} \quad n\in\mathbb{Z}$$

up to isom

(i) (vert. coric local LGP-monoids & assoc. Kummer theory)

$$\Psi_{cns}({}^{n,\circ}\mathfrak{F})_{\underline{v}}|_{\underline{t}} \sim {}^{\circ}\Psi({}^{n,\circ}\mathfrak{F})_{\underline{v}}|_{\underline{t}}$$

apply Prop 3.4 to)

$${}^{n,\circ}\mathcal{HT}^{D\text{-}\Theta^{\pm ell}NF} \xrightarrow[\text{Kummer alg'm}]{} \left(\underline{v}\in\underline{\mathbb{V}} \longmapsto {}^{(\infty)}\Psi_{LGP}\left({}^{n,\circ}\mathcal{HT}^{D\text{-}\Theta^{\pm ell}NF}\right)_{\underline{v}} \right)$$

w/ splittings (up to tor)

<u>vertically coric (local) LGP monoid</u>

($\mathbb{F}_l^{\times\pm}$-symm is compat. w/ log-link)

Kmm : $\Psi_{cns}({}^{n,m}\mathfrak{F}_{\underline{v}})|_{*} \xrightarrow{\sim} \Psi_{cns}({}^{n,\circ}\mathcal{D}_{\underline{v}})|_{*}$

Frob. ét

$\uparrow$ $\uparrow$

(Prop 3.4) (up to *). (Prop 3.5)

Kmm : ${}_{(\infty)}\Psi_{\mathcal{F}_{LGP}}({}^{n,m}\mathcal{HT}^{\Theta^{\pm ell}NF})|_{\succeq}$

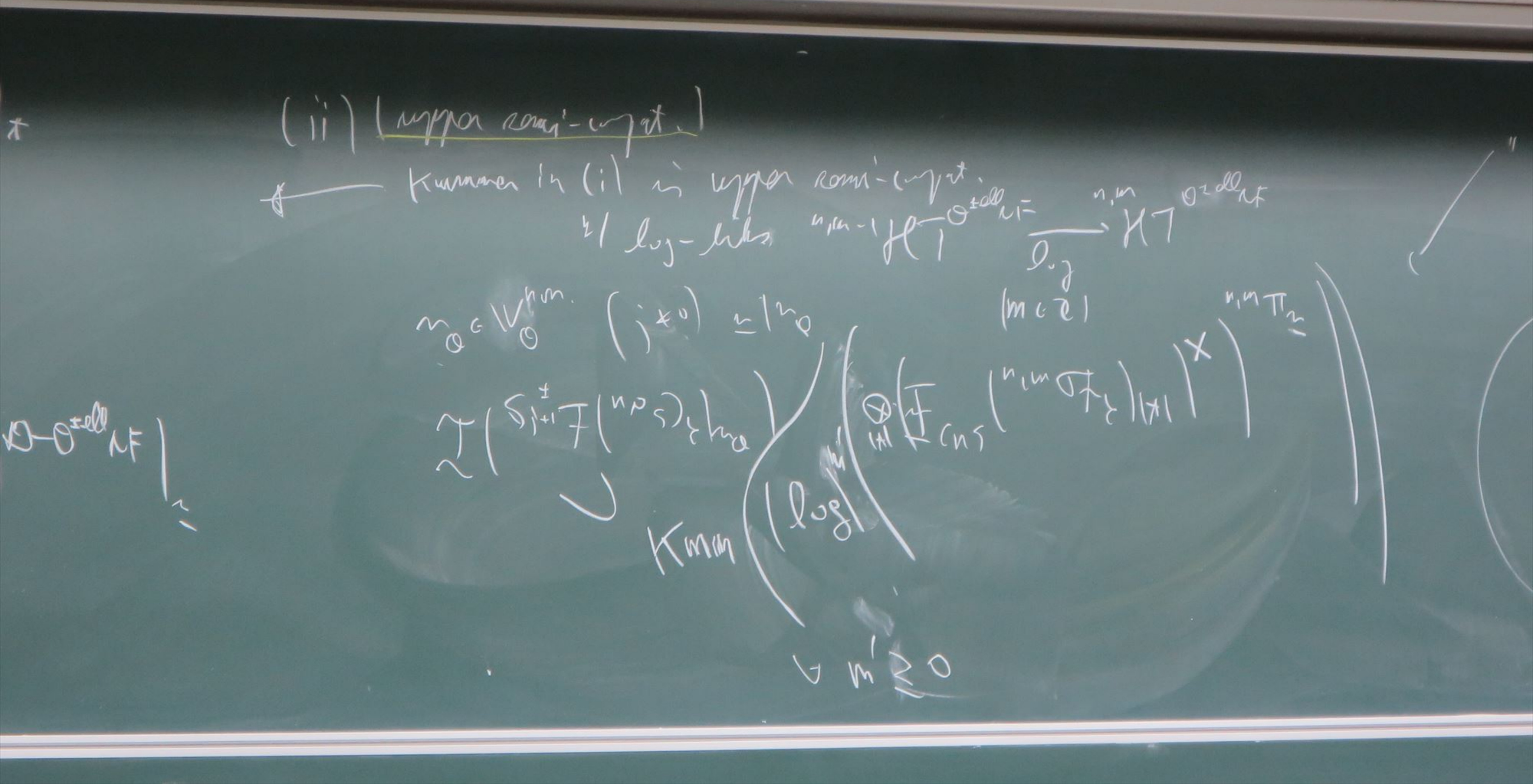
$\xrightarrow{\sim} {}_{(\infty)}\Psi_{LGP}({}^{n,\circ}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm ell}NF})|_{\succeq}$

(ii) (upper sem

$\longleftarrow$ Kummer i

$\mathcal{M}_{\Theta} \subseteq \mathbb{V}_{\Theta}$

$\mathcal{I}({}^{S_{j+}^{\pm}}$



*

(ii) (upper semi-compat.)

⟸ Kummer in (i) is upper semi-compat.

w/ log-links ${}^{n,m-1}\mathcal{H}\mathcal{T}^{\Theta^{\pm ell}NF} \xrightarrow{\log} {}^{n,m}\mathcal{H}\mathcal{T}^{\Theta^{\pm ell}NF}$

$(m \in \mathbb{Z})$

$n_{\Theta} \in \mathbb{V}_{\Theta}^{\mathrm{non}}$ $(j \neq 0)$ $\cong |^{n}\rho$

$\mathcal{D}\text{-}\Theta^{\pm ell}NF|$ $\cong$

$\underset{\sim}{\mathcal{I}}\left(S^{\pm}_{j+1}\mathcal{F}({}^{n}\rho_{S})_{\underline{v}_{\Theta}}\right)$ $\subseteq$ $\mathrm{Kmm}\left(\left|\log^{m'}\right|\left(\bigotimes_{|t|}\mathcal{F}_{(ns}\left({}^{n,m}\mathfrak{T}_{\underline{v}}\right)_{|t|}\right)^{\times}\right)^{{}^{n,m}\Pi_{\underline{v}}}$

$\forall\ m' \geq 0$

$\longrightarrow {}^{n,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}NF}$

∂

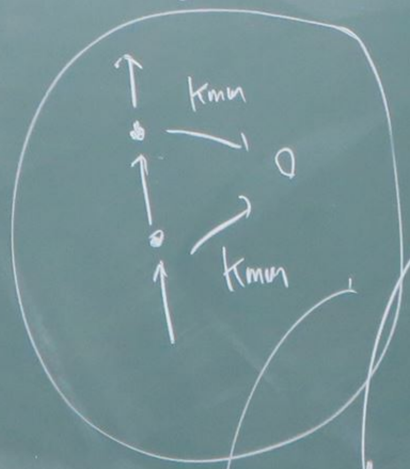
$e|$

$\sigma\mathcal{F}_{\xi})|_{|\times|})^{\times})^{n,m}\Pi_{\underline{v}}$

" $\mathcal{O}^{\times}_{\underline{v}} \subset \frac{1}{2p_{\underline{v}}}\log(\mathcal{O}^{\times}_{\underline{v}})$ "

$\cup$

$\log(\mathcal{O}^{\times}_{\underline{v}})$



non-interference

$\mathcal{I}^{\mathbb{Q}}(\mathcal{S}^{\pm}_{j+1,j}\ {}^{\mathcal{F}}(^{n,\circ}\mathfrak{S})_{\xi}|_{\underline{v}})$

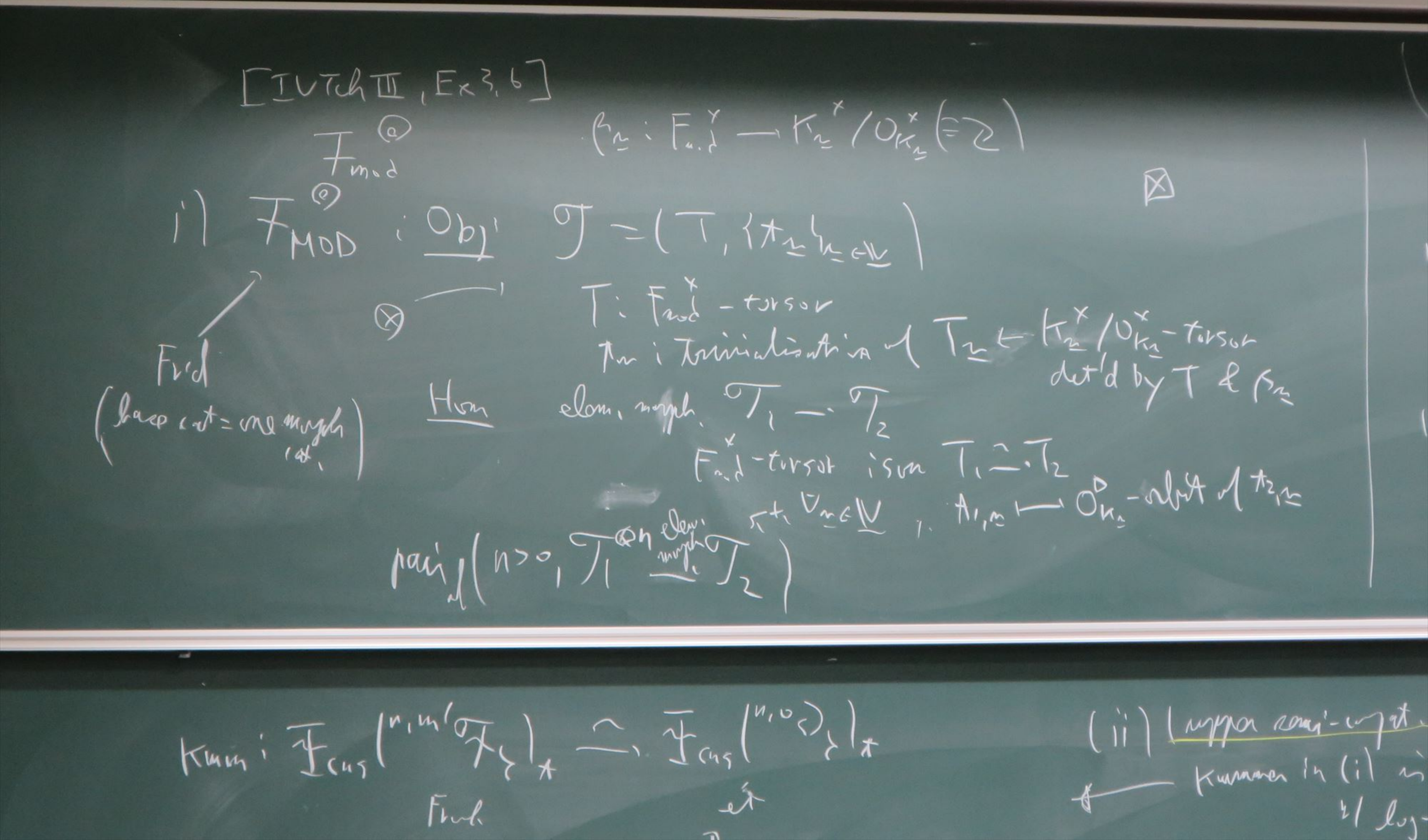
$(\subseteq \log(\mathcal{S}^{\pm}_{j+1,j};\ {}^{\mathcal{F}}(^{n,\circ}\mathfrak{S})_{\xi}|_{\underline{v}}))$

$\circlearrowright$

$(\omega)\ -\frac{1}{2\ell}\mathrm{Lap}(^{n,m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}NF}|_{\underline{v}}$

through $\mathrm{Kmm} \circ (\log)^{m}$ $m \geq 0$

totality of these actions



[IUTch III, Ex 3.6]

$\mathcal{F}_{\mathrm{mod}}^{\circledcirc}$ $\qquad \rho_{\underline{v}} : F_{\mathrm{mod}}^{\times} \longrightarrow K_{\underline{v}}^{\times}/\mathcal{O}_{K_{\underline{v}}}^{\times} (\cong \mathbb{Z})$

⊠

i) $\mathcal{F}_{\mathrm{MOD}}^{\circledcirc}$: Obj $\quad \mathcal{T} = (T, \{t_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$

⊗

Frd

(base cat = one morph cat.)

T : $F_{\mathrm{mod}}^{\times}$-torsor

$t_{\underline{v}}$: trivialisation of $T_{\underline{v}}$ ← $K_{\underline{v}}^{\times}/\mathcal{O}_{K_{\underline{v}}}^{\times}$-torsor det'd by T & $\rho_{\underline{v}}$

Hom $\quad$ elem. morph. $\mathcal{T}_1 \to \mathcal{T}_2$

$F_{\mathrm{mod}}^{\times}$-torsor isom $T_1 \xrightarrow{\sim} T_2$

s.t. $\forall \underline{v}\in\underline{\mathbb{V}}$, $t_{1,\underline{v}} \longmapsto \mathcal{O}_{K_{\underline{v}}}^{\triangleright}$-orbit of $t_{2,\underline{v}}$

pair of $(n>0, \mathcal{T}_1^{\otimes n} \xrightarrow{\text{elem. morph.}} \mathcal{T}_2)$

Kumm : $\mathcal{F}_{\mathrm{cusp}}({}^{n,m}\mathcal{F}_{\underline{v}})|_{\star} \xrightarrow{\sim} \mathcal{F}_{\mathrm{cusp}}({}^{n,\circ}\mathcal{D}_{\underline{v}})|_{\star}$

Frob. $\qquad$ ét

(ii) (upper semi-compat.

← Kummer in (i)

w/ log-

$\{\underline{q}^{j^2}\}_j \curvearrowright \widetilde{\mathcal{I}}^{\circ} \curvearrowleft (F^\times_{\mathrm{mod}})_{j}$

⊠

$^\times / O^\times_{K_{\underline{v}}}$-torsor

t'd by T & $K_{\underline{v}}$

$O^{\triangleright}_{K_{\underline{v}}}$-orbit of $t_{2,\underline{v}}$

4] ii) $\mathcal{F}^{\circledR}_{\mathrm{mod}}$

Frob'd

(base = [illegible])

<u>Obj</u> $\mathfrak{J} = \{J_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$

$J_{\underline{v}} \subset K_{\underline{v}}$ | fractional ideal $\underline{v}\in\underline{\mathbb{V}}^{\mathrm{non}}$

a positive real multiple of $O_{K_{\underline{v}}}$ $\underline{v}\in\underline{\mathbb{V}}^{\mathrm{arc}}$

s.t. $J_{\underline{v}} = O_{K_{\underline{v}}}$

for all but fin'ly many $\underline{v}$

$f \in F^\times_{\mathrm{mod}} \curvearrowright f\mathfrak{J} \in \mathrm{Ob}(\mathcal{F}^{\circledR}_{\mathrm{mod}})$

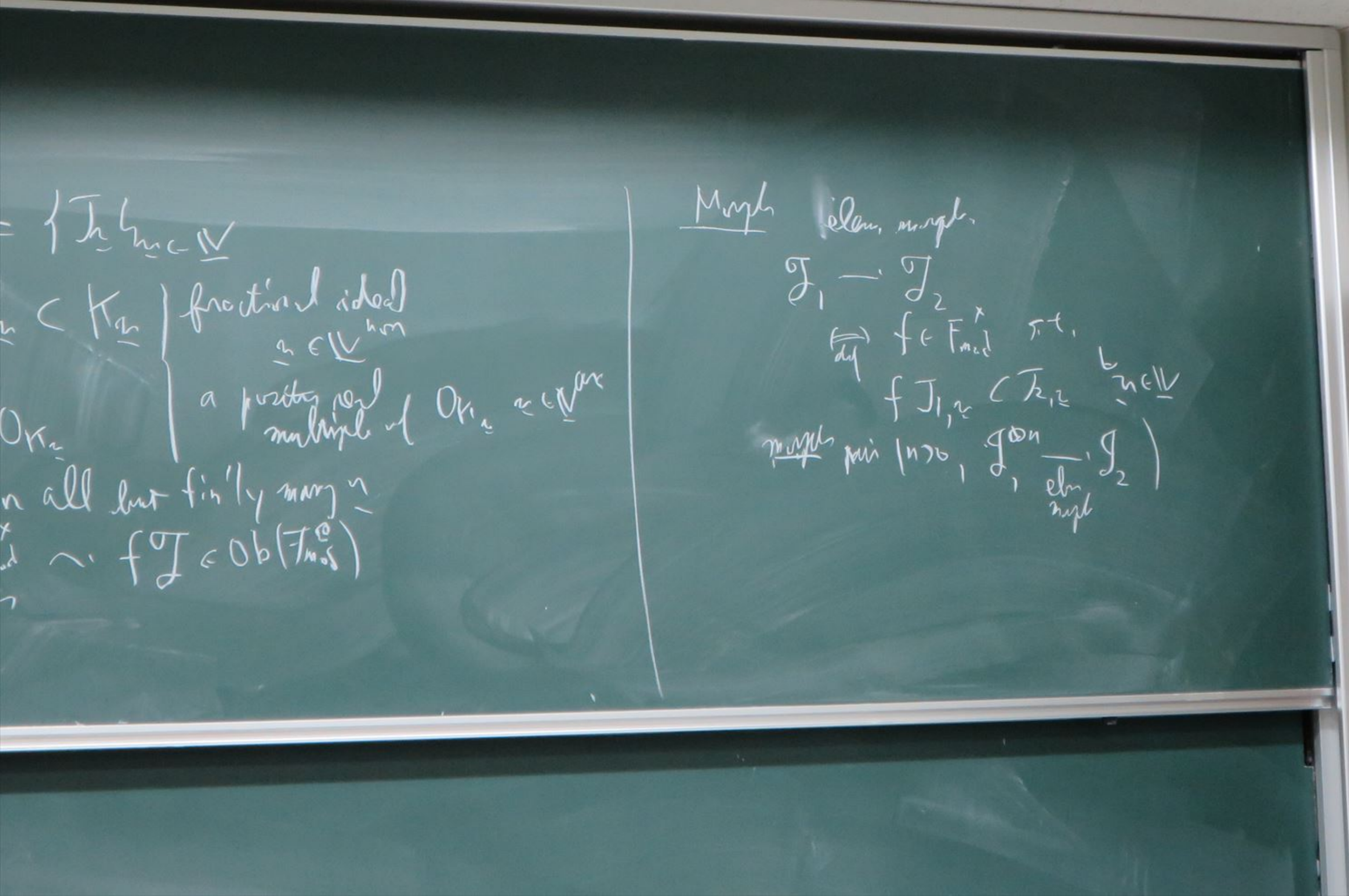
n $\mathfrak{J}^{\mathrm{an}}$

<u>Morph</u> elem

$\mathfrak{J}_1 \longrightarrow$

$\overset{\frown}{\mathrm{def}}$

<u>morph</u> pri



Frd

(base cat = one morph. cat.)

<u>Hom</u> elem. morph. $T_1 \to T_2$

$\mathcal{F}^{\times}_{\rm mod}$-torsor isom $T_1 \xrightarrow{\sim} T_2$ s.t. $\forall \underline{v} \in \underline{\mathbb{V}}$, $A_{1,\underline{v}} \longmapsto \mathcal{O}^{\triangleright}_{k_{\underline{v}}}$-orbit of $A_{2,\underline{v}}$

pairs $\left(n>0,\ T_1^{\otimes n} \xrightarrow[\text{morph.}]{\text{elem.}} T_2\right)$

$$\begin{tikzcd} \mathcal{F}^{\circledR}_{\rm mods} \arrow[r, "\sim"] & \mathcal{F}^{\circledR}_{\rm mod} \arrow[r, "\sim"] & \mathcal{F}^{\circledR}_{\rm MOD} \end{tikzcd}$$

$\boxplus$ $\qquad\qquad$ $\underbrace{\qquad\qquad}_{\mathcal{F}^{\times}_{\rm mod}\ \text{rig. str.}}$ $\qquad\qquad$ $\boxtimes$

↑ upper semi-compat. w/ log-link

↑ suited to estimate

↑ str. compat. w/ Θ-link

↑ difficult to compute

$\underline{q}^{j^2}$ Θ

$\underline{q}$ Θ

$\mu^{\log}$

$T_{\underline{n}} \leftarrow K_{\underline{n}}^{\times}/O_{K_{\underline{n}}}^{\times}$-torsor
det'd by T & $\kappa_{\underline{n}}$

$T_1 \overset{\sim}{-}\cdot T_2$

$A_{1,\underline{n}} \longmapsto O_{K_{\underline{n}}}^{\triangleright}$-orbit of $\pi_{2,\underline{n}}$

F'rd
(base = one morph'rat.)

s.t. $\mathcal{J}_{\underline{n}} = O_{K_{\underline{n}}}$
for all but fin'ly many $\underline{n}$

$\exists \in \mathbb{V}^{\text{non}}$

a positive real multiple

n $f \in F_{\text{mod}}^{\times} \rightsquigarrow f^{\sigma}\mathcal{J} \in \mathrm{Ob}(\mathcal{F}_{\text{mod}}^{\circledast})$
$\mathcal{J}^{\text{ori}}$

$q^{j^2}\Theta$

$\underline{q}\,\Theta$

$\mu^{\log}$

to

Θ_{const}

LCFT

env ~ gen

ch?

$\Theta \rightsquigarrow \underline{\log}(\ ) \rightsquigarrow \log.$

$0\text{-}g$

⊞ modules